Holographic Scaling in Newtonian Gravity

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Holographic Scaling in Newtonian Gravity

Emma Machado

A thesis presented for the Undergraduate Honors Program
Bachelor of Arts

Department of Natural Sciences, Physics
Assumption College
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Abstract

Many high school and college students are required to take physics, but few actually learn to discover the mysteries of the field, because they are too busy trying to memorize equations and solve "plug and chug" style problems. Looking into the calculations and equations of physics, a holography can be seen within the subject. This holography can be discovered and made accessible to general physics students, by studying the duality that exists between Electricity and Gravity. Furthermore, the concept of mass in physics ($M_{\text{ADM}}$) can be calculated, within this holography, for various black holes along with the orbits of massive and massless particles. This concept of holography allows physicists to break down complex problems. This research demonstrates how holography can translate information from 4D General Relativity to 2D Dilaton Gravity.
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1 Introduction

Physics is a universal field with complexity far beyond what people can imagine. It is where the unimaginable seems to somehow figure out a way to exist. In college, or even high school, many students are required to take a physics course. The introduction to any new topic or course can be overwhelming for students, and physics is no exception. However, if you put aside the feeling of, 'why do I need to take this course' and 'I guess I just need to memorize these problems to ace my exam,' there is actually a great wonder hidden beneath all the equations and formulas. This wonder is holography.

Holography is characterized by projecting something of many dimensions onto a surface of fewer dimensions. In 1993, Gerard 't Hooft proposed the holographic principle, which consists of two claims; the first states that a hologram can represent information of a given portion of space and the second states that the portion of space must for every Planck area, which is the smallest area in Quantum Gravity, have a degree of freedom [1]. Although Gerard 't Hooft was the first physicist to propose the concept of the holographic principle, the idea of holography actually dates back to the Ancient Greek philosopher Plato. "The Allegory of the Cave," by Plato foreshadows holography, by introducing the idea that the shadows the prisoners see in the cave represent reality [1]. Through this allegory, Plato is demonstrating that something with three dimensions can be projected onto a surface of two dimensions. This demonstration represents the definition of holography, which states that something of multiple dimensions can be projected onto a surface of fewer dimensions. In addition to Plato’s foreshadowing of holography, other scientists also discovered
holography before it was proposed in the context of physics in 1993. These scientists include Dennis Gabor, N. Bassov, A. Prokhorov, Emmett Leith, Juris Upatnieks, Dr. Yuri N. Denisyuk, Dr. Stephen A. Benton, Lloyd Cross, and Victor Komar. They used their knowledge of holography to make holograms of people and create other holographic images [2]. In physics, holography is not used to make holograms but rather to provide an understanding of complex problems and calculations. The information above about holograms is only included as an example of how the concept of holography can be used in a less theoretical and possibly more relatable way.

Before understanding how holography relates to physics, one needs to understand mass in Newton-Cartan Gravity. Newton-Cartan Gravity is a low energy approximation to Einstein’s full theory of Gravity, General Relativity [3]. The difference between Newton-Cartan Gravity and General Relativity is the types of problems that they are used to solve. Newton-Cartan Gravity is used to solve problems dealing with our solar system, planetary orbits, and space flight. On the other hand, General Relativity is used to solve problems dealing with high energy astrophysics, binary solar systems (solar systems with two suns), and black hole orbits, including a black hole-black hole orbit or a black hole-sun orbit. A black hole-black hole orbit is one black hole orbiting another and a black hole-sun orbit is a black hole orbiting a sun.

Einstein’s equations of General Relativity display the association between Gravity and the curvature of space. These equations developed by Einstein aid in the understanding and research of the theory of General Relativity [4]. The Einstein equations combine heat, temperature, and entropy and serve as
a basis from which the laws of mechanics for black holes were developed [5].

The other step to understanding mass in Newton-Cartan Gravity is defining mass. There are two types of mass: gravitational mass ($m_G$) and inertial mass ($m_I$). Gravitational mass is a conserved quantity associated with a time-translational symmetry in a physical theory. The conserved quantity comes from Noether’s Theorem, which states that for every symmetry, there exists a conserved quantity [6]. When there is a time-translational symmetry, this conserved quantity is mass. The mass of a thermodynamically closed system is a fixed value, since no mass can enter or escape the system. That means the mass of a particular closed system is the same today as it was 2,000 years ago and as it will be 2,000 years from now. This concept of fixed mass can be applied to black holes because they are thermodynamically closed systems.

The second type of mass is known as inertial mass. Inertial mass is an object’s property to resist change in its state of motion. The resistance that a car gives when the driver steps on the breaks at a red light or when the driver steps on the gas at a green light are functions of $m_I$. If a small car and a large delivery truck are stopped at a red light, the small car will move first when the light turns green. This is because there is less resistance to changing its state in motion. In other words, the small car has a smaller inertial mass than the large delivery truck. As far as we know, gravitational mass and inertial mass are equal up to 23 significant figures. This poses a problem for physicists because they do not know why they are equal.

The entire mass of a system is known as $M_{ADM}$. $M_{ADM}$ is the common designation for total mass in a four dimensional gravitational theory and was
first proposed by physicists Arnowitt, Deser, and Misner [6]. The value for $M_{ADM}$ can be determined by taking the limit of the rate at which the area that the mass is enclosed in is changing [7]. This thesis shows that taking the limit as the radius of a two sphere goes to infinity in which a four dimensional mass was contained. This allows the $M_{ADM}$ to be calculated and a holography to be observed. A two sphere is a sphere that has a blank surface that makes each point on the surface indistinguishable from any other point on the surface.

Newton’s laws are an important part of understanding why a book and a piece of paper, if dropped at the same time, will hit the floor at the same time. The two objects, regardless of their mass, will hit the floor at the same time. This phenomenon is because their acceleration does not depend upon their respective masses. This idea is explained through Newton’s gravitational force law and his first two laws of motion. Newton’s gravitational force law states that force is equal to Newton’s constant ($G$) times the mass of each object divided by the radius between them squared, $F^N = \frac{G m_1 m_2}{r^2}$. Newton’s first law of motion states that an object in motion tends to stay in motion, while an object at rest tends to stay at rest. This law describes the definition of inertial mass and ties into Newton’s second law of motion. The second law states that force is equal to mass times acceleration, $F = m_1 a$. The mass used in this law is inertial mass. When Newton’s force law and second law are set equal, we can find that the acceleration is $9.8 \text{ m/s}^2$, as long as the gravitational mass and inertial mass are equal as shown on the next page.
\[
\frac{GM_Em_G}{r^2} = F = m_1 a \\
\frac{GM_E}{r^2} = a \\
a = \frac{GM_E}{r^2} = \frac{GM_E}{R_E^2} = 9.81 \text{m/s}^2
\]

Dualities that exist in physics can be used to help understand concepts that less information is known about. For example, a duality that exists between entropy and temperature at the horizon of black holes can be used to better understand these characteristics of black holes [8]. These dualities allow physicists to explain complex theories with something that is more familiar to their audience. In addition, these dualities allow physicists to understand more about a concept based on the information that they already know. In order to understand Gravity, we can use the duality it has with a similar concept that many are familiar with, Electricity. Electricity has both positive and negative charges, an electric field, forces of attraction and repulsion, and its force law is Coulomb’s Law. Gravity has features like Electricity. These include charge, a gravitational field, and force. The force law for Gravity is Newton’s Gravitational Law. The charge of Gravity is positive and is also known as mass. The force of Gravity is attraction. After all, we do not all fly off the Earth, right?
<table>
<thead>
<tr>
<th>Charges</th>
<th>Electricity</th>
<th>Gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Positive and Negative</td>
<td>Positive gravitational charge (mass)</td>
</tr>
<tr>
<td>Field</td>
<td>Electric Field ($\vec{E}$)</td>
<td>Gravitational Field ($\vec{g}$)</td>
</tr>
<tr>
<td>Forces</td>
<td>Attraction and Repulsion</td>
<td>Attraction</td>
</tr>
<tr>
<td>Force Law</td>
<td>Coulomb’s Law: $F_c = \frac{kQq}{r^2}$</td>
<td>Newton’s Gravitational Law: $F_G = \frac{GMm}{r^2}$</td>
</tr>
</tbody>
</table>

Table 1: The known charges, field, forces, and force law of Electricity can be used to understand these same components of the theory of Gravity.
From this analogy between Electricity and Gravity, we can generate a duality map.

Duality Map between Electricity and Gravity

Constants: \( k \sim G \)

Charge: \( Q \sim M \) and \( q \sim m \)

Field: \( \vec{E} \sim \vec{g} \)

In this duality map, we can see that Coulomb’s constant and Newton’s constant are similar, charge in Electricity and mass in Gravity are similar, and the electric field and gravitational field are similar. This duality between Electricity and Gravity continues, but there is a problem. Forces are not fundamental, but fortunately, Gauss’ Law is fundamental, meaning that forces are dependent upon Gauss’ Law. It is from this law that Force Laws are derived. Thus, the term force is much weaker than Gauss’ Law. Gauss’ Law measures the field flow through a closed surface in relation to the source in the field. Gauss’ Law can be found for both Electricity and Gravity.
<table>
<thead>
<tr>
<th></th>
<th>Electricity</th>
<th>Gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’ Law</td>
<td>( \vec{E} \cdot \vec{A} = 4\pi k q_{\text{enclosed}} )</td>
<td>( \vec{g} \cdot \vec{A} = 4\pi G m_{\text{enclosed}} )</td>
</tr>
<tr>
<td></td>
<td>( \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\epsilon_0} )</td>
<td>( \oint \vec{g} \cdot d\vec{A} = \frac{m_{\text{enclosed}}}{\epsilon_0} )</td>
</tr>
</tbody>
</table>

Table 2: Using the duality map comparing the units of electricity to those of Gravity, Gauss’ law for Gravity can be written based off of Gauss’ law for Electricity.

Gauss’ law for Gravity can be solved to find the value of the mass enclosed in a given area, \( m_{\text{enclosed}} = \frac{1}{4\pi G} \vec{g} \cdot \vec{A} \). A four dimensional mass can be enclosed in a two sphere. Gauss’ Law can then be used to find out what portion of the mass is enclosed inside of the two sphere. To ensure that the entire four dimensional mass is enclosed in the two sphere, the radius of the sphere can be extended to infinity. Holography is this idea of containing an entire four dimensional mass in a two dimensional sphere. This thesis will continue with the theme of holography to show how \( M_{ADM} = \lim_{r \rightarrow \infty} \frac{\vec{g} \cdot \vec{A}}{4\pi G} \) in algebra, which can be expanded through calculus to \( M_{ADM} = \oint \Gamma_u \hat{n} dA \). It
will also show how you can go from four dimensional General Relativity to two dimensional Dilaton Gravity.

Black holes are complicated systems that exist in space that physicists are trying to understand through their research. The calculations conducted on the Schwarzschild black hole for finding the central charge of the symmetry algebra in 'Central charge for the Schwarzschild black hole' have an element of universality and can therefore be conducted in a similar manner on the other types of black holes [9]. A relationship exists between the central charge of the symmetry algebra and angular momentum. This relationship can be applied to the Reissner-Nordström, Kerr, and Kerr-Newman black holes. Aside from looking at the total mass, this thesis also looks at the orbit at the horizon of the four black holes mentioned above for both massive ($\epsilon = 1$) and massless ($\epsilon = 0$) particles. While learning about black holes, one can discover information about classical Dilaton Gravity. Gravity paired with a Dilaton field comes from the low energy explanation of dilatonic black holes in string theory. Thus, Dilaton Gravity has connections with black holes and string theory [10]. Furthermore, both the effective action and holography can be applied to find the near-extremal metric of the Kerr-Sen black hole. This focus on thermodynamics states that the Hawking temperature is proportional to the near-extremality parameter of the Kerr-Sen black hole [11].

The two dimensional Conformal Field Theory (CFT) is actually a hologram. Symmetries are an important element in physics, as they aid in understanding complex theories just like dualities. Holograms are fascinating because they exhibit a level of symmetry. This symmetry provides a greater
understanding of the string theory in the context of superstrings [12].

The proposal of the holographic principle by Gerard ’t Hooft in 1993 revolutionized components of modern physics. This principle has expanded the field of physics, allowing it to define things that were previously unknown. The holographic principle has caused physicists to rethink the components of physical systems. These elements include the entropy of systems and how to quantify the degrees of freedom of systems [13]. By having to rethink these characteristics of physical systems, it can be seen that the holographic principle has allowed physicists to look at problems in a new light. Furthermore, this principle is helping physicists to see contradictions that may exist in research, including those that may exist within crucial theories, such as Quantum Mechanics and General Relativity [13].

By reducing the dimensions, physicists are able to solve complicated calculations in an easier way. Rather than worrying about all the dimensions, a given problem can be solved by reducing these dimensions. Therefore, being able to reduce dimensions plays a significant role in expanding the discoveries in physics by making it possible to understand what is going on at a multi-dimensional level [14].

Little is known about Dilaton Gravity. There are some suggested ways in which Dilaton Gravity can be solved, based on symmetry. This type of Gravity allows physicists to understand the geometry of black holes. Therefore, before being able to work with or analyze Dilaton Gravity, it is crucial to understand key aspects of black holes. Since the Dilaton field is dimensionless, this allows physicists the ability to approach solving two dimensional Dilaton Gravity in
a variety of ways [15]. Currently, the information on two dimensional Dilaton Gravity is limited. Therefore, this thesis will provide more research to the concept of Dilaton Gravity in physics. Both the energy momentum tensor and Hawking radiation can be determined through two dimensional Dilaton Gravity combined with a scalar [16].

Therefore, this thesis will contribute to what is known about the holographic principle in the field of physics by demonstrating how this principle can be used to understand complex physics problems.
2 Background Information

2.1 Fundamental Forces

As mentioned earlier in the introduction, forces are not fundamental, which is why it is important to reference Gauss’ Law. However, there are four fundamental forces that exist. These forces are electromagnetic interactions, the strong nuclear force, the weak nuclear force, and Gravity.

<table>
<thead>
<tr>
<th>Fundamental Force</th>
<th>Strength</th>
<th>Range (meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electromagnetic Interactions</td>
<td>$\frac{1}{137}$</td>
<td>infinite</td>
</tr>
<tr>
<td>Strong Nuclear Force</td>
<td>1</td>
<td>$10^{-15}$</td>
</tr>
<tr>
<td>Weak Nuclear Force</td>
<td>$10^{-6}$</td>
<td>$10^{-18}$</td>
</tr>
<tr>
<td>Gravity</td>
<td>$6.0 \times 10^{-39}$</td>
<td>infinite</td>
</tr>
</tbody>
</table>

Table 3: The four fundamental forces that exist in physics are electromagnetic interactions, strong nuclear force, weak nuclear force, and Gravity. Each works on a range in meters, and the strength of each is quantified relative to the strength of the strong nuclear force [17].

Electromagnetic interactions occur in everyday life in electronics, including cell phones, computers, and televisions. According to electromagnetism, charges with the same charge repel each other, as seen with magnets. If the positive ends of two magnets face each other, the magnets will repel away from each other. Similarly, if the negative ends of two magnets face each other, the magnets will repel away from each other. Furthermore, if a positive end of one magnet and a negative end of another magnet face each other, the two will attract each other, or connect. This shows that like charges repel and opposite charges attract one another. Although not visible, the electromagnetic
force is actually what causes these interactions between magnets to occur. To visualize this force, the above mentioned experiment can be repeated with iron filings. Since electromagnetic interactions tell us that things can repel against or attract each other, then matter as we know it would not exist if this was the strongest force. Therefore, there must be some force stronger than electromagnetism that allows matter to exist.

This stronger force is known as the strong nuclear force. It is responsible for keeping the components of an atomic nucleus together and is involved in nuclear energy at power plants [17]. This force acts on a small range. If this force did not work on a small range, then every atom would be pushed together and we would all be squished together.

The weak nuclear force is a small range force involved in radiation. This force works in medical technology, such as PET scans, MRI, and X-rays. Physicists refer to electromagnetic interactions, the strong nuclear force, and the weak nuclear force as the Grand Unified Theory (GUT).

The fourth fundamental force, Gravity, is not included in the GUT, since physicists do not know how it acts in relation to the other three fundamental forces. The three fundamental forces that make up the GUT have force fields defined in spacetime. Gravity is unique, compared to these forces because rather than having a force field defined in spacetime, Gravity already exists as a characteristic of spacetime. The effects of Gravity that we encounter are a representation of the curvature of spacetime. Gravity is the weakest of all four of the fundamental forces and works on the large scale, keeping us on Earth. Gravity is also referred to as General Relativity.
2.2 General Relativity

General Relativity is a four dimensional theory of Gravity, which was developed by Einstein [3]. This theory can be understood through the use of tensors, tensor calculus, metrics, metric tensors, and the concept of curvature. They are highlighted in detail in the following subsections.
2.2.1 Tensors and Tensor Calculus

As explained above in the introduction, General Relativity is Einstein’s full theory of Gravity. In order to solve physics problems regarding Gravity, it is essential to use of tensors and tensor calculus. In order to understand what tensor calculus is we first must understand tensors. Tensors are indexed objects whose components transform through linear transformations or transform linearly through coordinate transformations, in other words they are an array of the coordinates in spacetime [3]. Interestingly, tensors are independent of coordinate systems, which allows them to be used in any coordinate system, making them universal. This property exhibited by tensors is unique since most other things in math are dependent upon coordinate systems and have to be transformed in order to move from one coordinate system to another. An example of this can be seen with the Jacobian matrix in the next subsection.

Since there is more than one type of tensor, physicists rank tensors based on their number of indices. In other words, the number of indices is the rank number of the tensor. A scalar is a number and therefore has no indices. This means that a scalar is a rank 0 tensor. A vector, which is used in many high school and undergraduate level physics courses and even some math courses, can be classified as a rank 1 tensor. Vectors only have one index. Both Gravity and Electricity are examples of rank 2 tensors. Each of these tensors involve two indices. The Riemann tensor, which is described in more detail in the next subsection, is an example of a rank 4 tensor.

Aside from being classified by rank, tensors can also be described as co-
variant, contravariant, antisymmetric, symmetric, and mixed. A covariant
tensor has indices in the subscript position, whereas a contravariant tensor
has indices in the superscript position. A mixed tensor is neither covariant
nor contravariant, as it has characteristics of both types of tensors. This mix-
ture of characteristics means that some of the indices are subscripts while the
others are superscripts. An antisymmetric tensor changes signs to positive or
negative when two indices swap places. On the other hand, a symmetric tensor
is one in which the sign remains the same when two indices swap places [3].

Vectors or tensors can be written in two dimensions or more. During
physics class, students generally learn about either two or three dimensional
vectors in cartesian coordinates. Two dimensional vectors have $x$ and $y$ com-
ponents, whereas three dimensional vectors have $x$, $y$, and $z$ components. In
physics and math, $x$ components are denoted by $\hat{i}$, $y$ components are denoted
by $\hat{j}$, and $z$ components are denoted by $\hat{k}$. The length of a vector is represented
by the following equation, $|<x_1, x_2, x_3>| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Take a function $f(x_1, x_2, x_3)$ where each $x$ can be parametrized as $x = x(t)$. Then the derivative of this function

$$df(x_1(t), x_2(t), x_3(t)) = \frac{df}{dt} = \frac{dx_1}{dt} \frac{df}{dx_1} + \frac{dx_2}{dt} \frac{df}{dx_2} + \frac{dx_3}{dt} \frac{df}{dx_3}$$

$$= \frac{dx_\mu}{dt} \frac{df}{dx_\mu} \text{ where } 1 \leq \mu \leq 3$$

Above, the $\frac{dx_\mu}{dt}$ is the vector, can also be written as $A^\mu$ and $\frac{df}{dx_\mu}$ is the basis
of the function and can be rewritten as $\partial \mu(f)$. Therefore, $\frac{dx_\mu}{dt} \frac{df}{dx_\mu} = A^\mu \partial \mu(f)$
where $1 \leq \mu \leq 3$. 

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It is important to understand what tensors are and how to perform calculations using them. Knowing how to perform calculations using tensors comes from the branch of math known as multivariable calculus, which is also known as tensor calculus or vector calculus.

There are two main calculations that can be made using tensors. The first is the cross product. The cross product between two vectors results in a vector and uses a matrix. It is important to note that the cross product is not commutative, meaning that crossing a tensor \( x \) with a tensor \( y \) is not guaranteed to produce the same result as crossing the tensor \( y \) with the tensor \( x \). A tensor can be written as \(< x_1, x_2, x_3 >\) or as \( x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k} \) where \( \hat{i}, \hat{j}, \) and \( \hat{k} \) represent unit vectors in the direction \( x, y, \) and \( z, \) respectively. An example of a cross product of two tensors can be seen on the next page.
Tensor $\mathbf{x} = \langle x_1, x_2, x_3 \rangle = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}

Tensor $\mathbf{y} = \langle y_1, y_2, y_3 \rangle = y_1 \hat{i} + y_2 \hat{j} + y_3 \hat{k}$

\[
\mathbf{x} \times \mathbf{y} = \text{determinant} \begin{vmatrix} 
\hat{i} & \hat{j} & \hat{k} \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 
\end{vmatrix} = 
\begin{vmatrix} 
x_2 & x_3 & \hat{i} \\
y_2 & y_3 & x_1 \\
x_1 & x_2 & y_1 
\end{vmatrix} - 
\begin{vmatrix} 
x_1 & x_3 & \hat{j} \\
y_1 & y_3 & x_2 \\
x_1 & x_2 & y_1 
\end{vmatrix} \hat{k} 
= (x_2 y_3 - x_3 y_2) \hat{i} - (x_1 y_3 - x_3 y_1) \hat{j} + (x_1 y_2 - x_2 y_1) \hat{k} 
= \langle x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1 \rangle
\]

\[
\mathbf{y} \times \mathbf{x} = \text{determinant} \begin{vmatrix} 
\hat{i} & \hat{j} & \hat{k} \\
y_1 & y_2 & y_3 \\
x_1 & x_2 & x_3 
\end{vmatrix} = 
\begin{vmatrix} 
y_2 & y_3 & \hat{i} \\
x_2 & x_3 & x_1 \\
x_1 & x_2 & y_1 
\end{vmatrix} - 
\begin{vmatrix} 
y_1 & y_3 & \hat{j} \\
x_1 & x_3 & x_2 \\
x_1 & x_2 & y_1 
\end{vmatrix} \hat{k} 
= (y_2 x_3 - y_3 x_2) \hat{i} - (y_1 x_3 - y_3 x_1) \hat{j} + (y_1 x_2 - y_2 x_1) \hat{k} 
= \langle y_2 x_3 - y_3 x_2, -y_1 x_3 + y_3 x_1, y_1 x_2 - y_2 x_1 \rangle
\]

\[
\mathbf{x} \times \mathbf{y} = - (\mathbf{y} \times \mathbf{x}) \\
\mathbf{y} \times \mathbf{x} = - (\mathbf{x} \times \mathbf{y})
\]

The example above visually represents the calculation of a cross product between two tensors and demonstrates the fact that this calculation is not
commutative. First, the tensors are written in a matrix with the first row being $\hat{i}$, $\hat{j}$, and $\hat{k}$, the second row being the values of $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively of the first tensor, and the third row being $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively of the second tensor. The resulting answer is a tensor that is perpendicular to both of the original two tensors. If the cross product between two tensors is equal to zero, then those two tensors are either parallel or antiparallel to each other. Parallel tensors have the same magnitude and direction but are at different positions. Antiparallel tensors have the same magnitude but have opposite directions. The cross product actually gives us the area of a parallelogram that can be produced by drawing the two vectors in tip to tail formation. This means that the tail of one vector is drawn at the tip of the other vector, while maintaining the direction and magnitude of each. Therefore, the resulting vector from the cross product is actually a measure of the area between the two vectors. In other words, area is a vector.

The other calculation that can be conducted with two tensors is a dot product. Unlike the cross product, the dot product of two tensors is commutative. The dot product results in a scalar since there is no free index. A scalar is a magnitude with no direction. A free index of a tensor is an index that is not repeated or summed over. The dot product of two tensors can be seen in the example on the next page.
Tensor $\vec{x}$: $\vec{x} = <x_1, x_2, x_3> = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

Tensor $\vec{y}$: $\vec{y} = <y_1, y_2, y_3> = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$

$$\vec{x} \cdot \vec{y} = <x_1, x_2, x_3> \cdot <y_1, y_2, y_3>$$

$$= (x_1\hat{i} + x_2\hat{j} + x_3\hat{k}) \cdot (y_1\hat{i} + y_2\hat{j} + y_3\hat{k})$$

combining like terms

$$= x_1y_1 + x_2y_2 + x_3y_3$$

To calculate the dot product multiply the first components of the two vectors together, adding the product of the second components of the vectors, and then adding the product of the third components of the vectors.

Generally, calculus courses in high school and at the undergraduate level deal with two dimensions. In other words, $(x, y)$ in cartesian or rectangular coordinates and $(r, \theta)$ in polar coordinates. The branch of calculus concerned with these two dimensional coordinate systems is single variable calculus. In cartesian coordinates, this would be having an equation $y$ in terms of $x$ for which you can find both the derivative and antiderivative. Since you are only differentiating or integrating $y$ with respect to $x$, you are only concerned with one variable.

Tensor calculus allows us to expand single variable calculus into multivariable calculus. Through tensor calculus, derivatives and antiderivatives can be taken of multivariable functions. The process of finding derivatives and antiderivatives is the same except you have to pay attention to which variable you are taking the derivative or antiderivative with respect to. Since a multiple
variable function has more than one derivative, we can find the derivative with respect to one of the variables. To better understand the connections between single variable calculus and multivariable calculus, see the table below.

<table>
<thead>
<tr>
<th></th>
<th>Single Variable Calculus</th>
<th>Multivariable Calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivatives</td>
<td>( y = mx + b )</td>
<td>( z = x^2 + xy + y^3 - b )</td>
</tr>
<tr>
<td></td>
<td>where ( m ) and ( b ) are constants</td>
<td>where ( b ) is a constant</td>
</tr>
<tr>
<td></td>
<td>( \frac{dy}{dx} = m )</td>
<td>( \frac{dz}{dx} = 2x + y )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{dz}{dy} = x + 3y^2 )</td>
</tr>
<tr>
<td>Integrals</td>
<td>( \int_{a}^{b} x^2 , dx )</td>
<td>( \int_{a}^{b} \int_{u}^{v} (x^2 + xy - y^3) , dy , dx )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = \int_{a}^{b} x^2 , dx + \frac{xy^2}{2} - \frac{y^4}{4} \bigg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = \int_{a}^{b} ((vx^2 + \frac{xv^2}{2} - \frac{v^4}{4}) - (ux^2 + \frac{ux^2}{2} - \frac{u^4}{4})) , dx )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>At this point the integral is evaluated</td>
</tr>
<tr>
<td></td>
<td></td>
<td>the same as a single variable integral.</td>
</tr>
</tbody>
</table>

Table 4: Taking both derivatives and integrals in multivariable calculus is parallel to that in single variable calculus. The only difference is that the other variables in the function are treated as constants when the derivative or integral is taken with respect to a particular variable.

The derivatives taken with respect to a variable in a multiple variable function are called partial derivatives. A partial derivative is denoted as \( \partial x \) instead of \( dx \), and is computed similarly to that of a regular derivative. However, it is a bit more complicated, since there are other variables in the function. To take the partial derivative of a function, take the derivative of the desired variable while holding the others constant. This means that you look at the
other variables in the function as numbers. For example, if the function was
\[ z = 2xy + 7y - 4x^2 + 3 \] and you wanted to find the \( \partial x \), you would view the \( y \)
in the problem as a constant resulting in \( \frac{\partial z}{\partial x} = 2y - 8x \).

Similar to differentiation, integrating a multiple variable function is anal-
ogous to integrating a single variable function. To integrate a multiple variable
function, the same rules apply from single variable calculus. To take the in-
tegral with respect to one variable, hold all the other variables in the integral
constant. This integral can be referred to as a partial integral. For both differ-
entiation and integration of multiple variable functions, it is important to note
what variable you are integrating or differentiating with respect to and view
all the other variables in the function as constants. Below is a full example of
integration of a multiple variable function.

\[
\int_0^1 \int_0^1 (5xy) \, dx \, dy = \int_0^1 \frac{5x^2 y}{2} \bigg|_0^1 \, dy
\]
\[
= \int_0^1 \left( \frac{5(1)^2 y}{2} - \frac{5(0)^2 y}{2} \right) \, dy
\]
\[
= \int_0^1 \frac{5y}{2} \, dy
\]
\[
= \left. \frac{5y^2}{4} \right|_0^1
\]
\[
= \frac{5(1)^2}{4} - \frac{5(0)^2}{4}
\]
\[
= \frac{5}{4}
\]

It is important to work from the inside to the outside when taking multiple
integrals. The innermost integral corresponds to the inner most \( d \). Similarly,
the outermost integral corresponds to the outermost \( d \).
Early on in their math education, students learn about the Pythagorean theorem, $x^2 + y^2 = z^2$ where $x$ and $y$ are the sides of the triangle that form a right angle, and $z$ is the hypotenuse. This theorem comes in handy when working with vectors. This crucial theorem allows physicists to break up vectors into their $x$ and $y$ components. If we place this triangle on a flat surface, then there is only one way to draw the hypotenuse. However, if this triangle is placed on a curved surface, then there are two ways to draw the hypotenuse. One way is drawing the flat hypotenuse that is the result from solving the Pythagorean theorem. The other follows the curve of the surface. Einstein wondered how to calculate this distance that traveled the curve. As a result, measuring the distance traveled by the curve uses Gravity. The flat surface has a spacial flat gravitational field, meaning that there is no energy or mass. Extending the Pythagorean Theorem allows us to see that Gravity is really a metric. Einstein realized that the difference between these two lines can be measured by the gravitational field.

The Maxwell equations displayed on the next two pages are well-known equations in physics that involve the tensor calculations mentioned in this section.
Gauss' Law for Electricity

\[ \vec{E} \cdot \vec{A}_\odot = \frac{q_{\text{enclosed}}}{\epsilon_0} \]
\[ \vec{E} \cdot \vec{A}_\odot = \oint \vec{E} d\vec{A} \]
\[ q_{\text{enclosed}} = \int \rho dV \]
\[ \oint \vec{E} d\vec{A} = \frac{1}{\epsilon_0} \int \rho dV \]
\[ \oint \vec{E} d\vec{A} = \int (\nabla \cdot \vec{E}) dV \]
\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \]

Lenz's Law

\[ \vec{B} \cdot \vec{L}_\odot = \mu_0 I \]
\[ \vec{B} \cdot \vec{L}_\odot = \oint \vec{B} d\vec{L} \]
\[ \oint \vec{B} d\vec{L} = \mu_0 I \]
\[ I = \oint \vec{J} d\vec{A} \]
\[ \oint \vec{B} d\vec{L} = \int (\nabla \times \vec{B}) d\vec{A} \]
\[ \int (\nabla \times \vec{B}) d\vec{A} = \mu_0 \oint \vec{J} d\vec{A} \]
\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]

Faraday's Law

\[ \vec{E} \cdot \vec{L}_\odot = -\frac{\delta (\vec{B} \cdot \vec{A})}{\delta t} \]
\[ \vec{E} \cdot \vec{L}_\odot = \oint \vec{E} d\vec{L} \]
\[ \oint \vec{E} d\vec{L} = \int (\nabla \times \vec{E}) d\vec{A} \]
\[ -\frac{\delta (\vec{B} \cdot \vec{A})}{\delta t} = -\frac{\partial}{\partial t} \oint \vec{B} d\vec{A} \]
\[ \int (\nabla \times \vec{E}) d\vec{A} = -\frac{\partial}{\partial t} \oint \vec{B} d\vec{A} \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
Gauss' Law for Magnetism

\[
\vec{B} \cdot \vec{A}_\odot = 0
\]
\[
\vec{B} \cdot \vec{A}_\odot = \oint \vec{B} d\vec{A}
\]
\[
\oint \vec{B} d\vec{A} = \int (\nabla \cdot \vec{B}) dV
\]
\[
\int (\nabla \cdot \vec{B}) dV = 0
\]
\[
0 = \int dV
\]
\[
\int (\nabla \cdot \vec{B}) dV = \int dV
\]
\[
\nabla \cdot \vec{B} = 0
\]
2.2.2 Metrics and Metric Tensors

The calculations in physics expand past the use of tensors and tensor calculus into metrics. In tensor calculus, there are more than one type of coordinate system that can be used to describe a problem. The two used in this research were cartesian coordinates and spherical coordinates. Both were in three dimensions and then expanded into four dimensions by including time. Many are familiar with cartesian coordinates also known as rectangular coordinates. In two dimensions, these coordinates are \((x, y)\), and in three dimensions, they are \((x, y, z)\). These coordinates are the basis of the coordinate systems, and all other coordinate systems are simply a transformation from this base.

\[
\begin{align*}
\text{Figure 1: Cartesian coordinates in three dimensions are } & (x, y, z) \text{ (left) and } \\
\text{spherical coordinates in three dimensions are } & (r, \theta, \phi) \text{ (right).}
\end{align*}
\]

In Cartesian coordinates for a flat surface with \((x, y)\), \(ds^2 = dx^2 + dy^2 = \left(dx \ dy\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}\). The gravitational metric for the flat surface is
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]. This metric is also be known as the identity matrix, since there are
only values on the diagonal from the top left to the bottom right and each of
these values is 1. It is called the identity matrix for its unique ability to be mul-
tiple to another matrix, such that the original matrix remains unchanged. This
same property also exists in polar coordinates. 
\[
ds^2 = r^2 d\theta^2 + r^2 \sin \theta^2 d\phi^2 = (d\theta \ d\phi) \cdot \begin{pmatrix}
r^2 & 0 \\
0 & r^2 \sin \theta^2
\end{pmatrix} \cdot \begin{pmatrix}
d\theta \\
d\phi
\end{pmatrix}.
\]
The gravitational metric is
\[
\begin{pmatrix}
r^2 & 0 \\
0 & r^2 \sin \theta^2
\end{pmatrix}.
\]
The transformation from cartesian coordinates to another coordinate sys-
tem is given by the Jacobian metric. The Jacobian metric is found by taking
the partial derivatives of \(x, y,\) and \(z\) with respect to \(r, \theta,\) and \(\phi.\) This produces
a 3x3 matrix. In order to find the transformation, the determinant of this
matrix must be taken. Since the Jacobian metric provides a transformation
from one coordinate system to another, it is also commonly referred to as the
Jacobian transformation.

A metric is a 4x4 matrix, which is a matrix with four columns and four
rows. Gravity in General Relativity can be represented as the gravitational
metric, which is a rank two symmetric covariant tensor. By taking the inverse
of each element in a metric, the resulting metric will be the inverse metric of
the initial metric. By multiplying a metric and its inverse, the resulting metric
should be the identity matrix.

From the gravitational field described above, the Christoffel symbol, which
comes into play in the covariant derivative, can be calculated. The Christoff-
kel symbol cancels the non-linear portion of this derivative. The covariant
derivative is equal to the derivative plus the Christoffel symbol. Gravity is covariantly constant, meaning that the covariant derivative of Gravity is equal to zero \[3\].

A flat surface has derivatives with respect to \(x\) and \(y\), such that \(dxdy - dydx = 0\). This does not apply to surfaces that have curvature. Instead, we get a solution, which is known as the Riemann curvature tensor. The Riemann curvature tensor is found by combining Christoffel symbols. The equation for the Riemann tensor is written below.

\[
R^\alpha_{\sigma\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \sum_{\gamma=1}^{n}(\Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\sigma} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\mu\sigma}) \quad \text{where } n \text{ is the number of dimensions.}
\]

Below is an example of how to use the Riemann equation using the two sphere. Since the two sphere is two dimensions, each of the four indices of the Riemann tensor, \(\alpha, \sigma, \mu, \) and \(\nu\), can either have a value of 1 or 2.

\[
R^1_{111} = \partial_\theta \Gamma^1_{11} - \partial_\theta \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{11} - \Gamma^1_{11} \Gamma^1_{11} + \Gamma^1_{12} \Gamma^2_{11} - \Gamma^1_{12} \Gamma^2_{11} = 0
\]

\[
R^1_{112} = \partial_\theta \Gamma^1_{21} - \partial_\theta \Gamma^1_{21} + \Gamma^1_{21} \Gamma^1_{11} - \Gamma^1_{21} \Gamma^1_{11} + \Gamma^1_{12} \Gamma^2_{21} - \Gamma^1_{12} \Gamma^2_{21} = 0
\]

\[
R^1_{121} = \partial_\phi \Gamma^1_{11} - \partial_\theta \Gamma^1_{21} + \Gamma^1_{11} \Gamma^1_{21} - \Gamma^1_{11} \Gamma^1_{21} + \Gamma^1_{12} \Gamma^2_{11} - \Gamma^1_{12} \Gamma^2_{11} = 0
\]

\[
R^1_{122} = \partial_\phi \Gamma^1_{21} - \partial_\phi \Gamma^1_{21} + \Gamma^1_{21} \Gamma^1_{21} - \Gamma^1_{21} \Gamma^1_{21} + \Gamma^1_{12} \Gamma^2_{21} - \Gamma^1_{12} \Gamma^2_{21} = 0
\]

\[
R^1_{211} = \partial_\theta \Gamma^1_{12} - \partial_\theta \Gamma^1_{12} + \Gamma^1_{11} \Gamma^1_{12} - \Gamma^1_{11} \Gamma^1_{12} + \Gamma^1_{12} \Gamma^2_{12} - \Gamma^1_{12} \Gamma^2_{12} = 0
\]
\( R_{212}^1 = \partial \theta \Gamma_{22}^1 - \partial \phi \Gamma_{12}^1 + \Gamma_{11}^1 \Gamma_{22}^1 - \Gamma_{21}^1 \Gamma_{12}^1 + \Gamma_{12}^1 \Gamma_{22}^2 - \Gamma_{22}^1 \Gamma_{12}^2 \\
= \partial \theta (-\sin \theta \cos \theta) - (-\sin \theta \cos \theta \cot \theta) \\
= -\sin \theta (-\sin \theta) + \cos \theta (-\cos \theta) + \cos \theta^2 \\
= -\cos \theta^2 + \sin \theta^2 + \cos \theta^2 \\
= \sin \theta^2 \\

\)

\( R_{221}^1 = \partial \phi \Gamma_{12}^1 - \partial \theta \Gamma_{22}^1 + \Gamma_{21}^1 \Gamma_{12}^1 - \Gamma_{11}^1 \Gamma_{12}^1 + \Gamma_{22}^1 \Gamma_{12}^2 - \Gamma_{12}^2 \Gamma_{22}^2 \\
= -\partial \theta (-\sin \theta \cos \theta) + (-\sin \theta \cos \theta \cot \theta) \\
= -(\sin \theta^2 - \cos \theta^2) - \cos \theta^2 \\
= -\sin \theta^2 + \cos \theta^2 - \cos \theta^2 \\
= -\sin \theta^2 \\

\)

\( R_{222}^1 = \partial \phi \Gamma_{22}^1 - \partial \phi \Gamma_{22}^1 + \Gamma_{21}^1 \Gamma_{22}^1 - \Gamma_{21}^1 \Gamma_{22}^2 + \Gamma_{22}^1 \Gamma_{22}^2 - \Gamma_{22}^2 \Gamma_{22}^2 = 0 \\

R_{111}^2 = \partial \theta \Gamma_{11}^2 - \partial \theta \Gamma_{11}^2 + \Gamma_{11}^2 \Gamma_{11}^1 - \Gamma_{11}^2 \Gamma_{11}^1 + \Gamma_{12}^2 \Gamma_{11}^1 - \Gamma_{12}^2 \Gamma_{11}^1 = 0
\[ R_{112}^2 = \partial \theta \Gamma_{21}^2 - \partial \phi \Gamma_{11}^2 + \Gamma_{11}^2 \Gamma_{21}^1 - \Gamma_{21}^2 \Gamma_{11}^1 + \Gamma_{12}^2 \Gamma_{21}^2 - \Gamma_{22}^2 \Gamma_{11}^1 \]
\[ = \partial \theta \cot \theta + \cot \theta^2 \]
\[ = -\csc \theta^2 + \cot \theta^2 \]
\[ = -\frac{1 - \cos \theta^2}{\sin \theta^2} \]
\[ = \frac{-\sin \theta^2}{\sin \theta^2} = -1 \]

\[ R_{121}^2 = \partial \phi \Gamma_{11}^2 - \partial \theta \Gamma_{21}^2 + \Gamma_{21}^2 \Gamma_{11}^1 - \Gamma_{11}^2 \Gamma_{21}^1 + \Gamma_{22}^2 \Gamma_{11}^1 - \Gamma_{12}^2 \Gamma_{21}^2 \]
\[ = -\partial \theta \cot \theta - \cot \theta^2 \]
\[ = \csc \theta^2 - \cot \theta^2 \]
\[ = \frac{1 - \cos \theta^2}{\sin \theta^2} \]
\[ = \frac{-\sin \theta^2}{\sin \theta^2} = 1 \]

\[ R_{122}^2 = \partial \phi \Gamma_{21}^2 - \partial \phi \Gamma_{21}^2 + \Gamma_{21}^2 \Gamma_{12}^1 - \Gamma_{21}^2 \Gamma_{12}^1 + \Gamma_{22}^2 \Gamma_{12}^2 - \Gamma_{22}^2 \Gamma_{21}^2 = 0 \]

\[ R_{211}^2 = \partial \theta \Gamma_{21}^2 - \partial \theta \Gamma_{21}^2 + \Gamma_{11}^2 \Gamma_{21}^1 - \Gamma_{11}^2 \Gamma_{21}^1 + \Gamma_{12}^2 \Gamma_{21}^2 - \Gamma_{12}^2 \Gamma_{21}^2 = 0 \]

\[ R_{212}^2 = \partial \theta \Gamma_{22}^2 - \partial \phi \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^1 - \Gamma_{12}^2 \Gamma_{12}^1 + \Gamma_{12}^2 \Gamma_{22}^2 - \Gamma_{22}^2 \Gamma_{12}^2 = 0 \]

\[ R_{221}^2 = \partial \phi \Gamma_{12}^2 - \partial \theta \Gamma_{22}^2 + \Gamma_{21}^2 \Gamma_{12}^1 - \Gamma_{22}^2 \Gamma_{12}^1 + \Gamma_{22}^2 \Gamma_{22}^2 - \Gamma_{22}^2 \Gamma_{12}^2 = 0 \]
The Riemann curvature tensor can be collapsed into what is known as the Ricci tensor. The Ricci tensor is a variant of the Riemann curvature tensor that relates to Gravity. The equation for the Ricci tensor is written below.

\[ R_{\mu\nu} = R^\alpha_{\sigma\mu\nu} = \partial_{\mu} \Gamma^\alpha_{\nu\sigma} - \partial_{\nu} \Gamma^\alpha_{\mu\sigma} + \sum_{\gamma=1}^{n} (\Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\sigma} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\mu\sigma}) \]

The Ricci scalar can be found by multiplying the inverse of the gravitational metric by the Ricci tensor. The equation for the Ricci scalar is

\[ R = g^{\mu\nu} R_{\mu\nu} \]

The equations above demonstrate the fact that the calculations for Riemann tensor, Ricci tensor, and Ricci scalar depend on each other, since the Riemann tensor is used in calculating the Ricci tensor, which is used in calculating the Ricci scalar.
When students are taught physics, one of the first concepts they learn about are forces. However, in reality, these forces do not actually exist. Instead these forces represent what is more commonly known to physicists as curvature. Curvature is the way that spacetime warps around objects. In a physics class at the high school or undergraduate level, it is easier to teach students about forces even though they are not fundamental in the field of physics. This teaching method is used to introduce students to physics without overwhelming them with its complexity.

As mentioned above, curvature is the way that spacetime warps around objects. To model this concept, take a look at the Sun. In many science classes, students are taught that the planets orbit the Sun. This phenomenon occurs due to the curvature that exists in spacetime, rather than through forces between the Sun and each planet. Many physicists have demonstrated this using a sheet of spandex and solid balls of different masses. When a ball is placed in the middle of the sheet of spandex rather than staying stiff and straight, the spandex curves around the ball. When other balls are pushed around the spandex and let go, they revolve around the ball in the center of the spandex. Now imagine this sheet of spandex as spacetime. Just like the spandex, spacetime is a flexible material that moves and curves around the objects within it.

Understanding what curvature means and looks like is an important component of understanding how physicists can use curvature in their problems. In either high school or college level physics, students learn about the position,
velocity, and acceleration of objects. If the class these students are taking is calculus based, they will learn that the velocity is the derivative of the position, since velocity is the rate of change of the position and tangent to the given position. Acceleration is the rate of change of the tangent, which can also be referred to as curvature.

Using a marker and a globe, one can see the effects of curvature and why it is important. If you point the marker south at the equator and move it in a triangle (right, up, and back down to the starting point) as seen in the images on the next page, the marker will now be facing southeast.
Figure 2: The marker was moved on the globe to symbolize how a vector changes direction as its position is moved around a curved surface.

Now imagine that marker as a vector. Along a curved surface, the vector changes based on the way it is moved, whereas on a flat surface, the vector would remain the same. The way in which a vector changes on a surface depends upon the curvature of that surface. This dependence is why the vector changes on the curved surface of the globe but remains the same on the flat surface of the table, as shown on the next page.
Figure 3: The experiment with the marker on the globe was repeated with the marker on a table to demonstrate how the direction of the vector is maintained even when the position is altered.

From these two examples of vector movement, it can be determined that the surface a vector is on affects the direction of the vector, as the position of the vector is altered. Looking at a globe, we can see that the surface is not entirely flat and that there is some curvature to it. Therefore, when we move the marker representing a vector, we are able to see that although the marker starts and ends at the same point, its direction has changed. However, when the marker was moved similarly on the table, its direction did not change. Therefore, there has to be a difference in the surface of the globe and the table. This difference between the surfaces is what physicists call curvature.
The Riemann tensor provides us with the measure of curvature. This tensor can even be condensed into what is known as the Ricci tensor. The Ricci tensor is a rank two tensor, whereas the Riemann tensor is a rank four tensor. Therefore, this condensation causes calculations to be easier. The Ricci scalar is known as the curvature scalar. In addition, the Christoffel symbol shows the connection between surface curvature and how the vector changes.

Young’s theorem shows that $\partial x \partial y - \partial y \partial x = 0$ is true when there is no curvature present [6]. This theorem works under linear conditions when the surface is flat but fails when the surface has curvature. This restriction is explained through the way that the direction of the vector changes as it is moved along the curved surface of the globe, but remains the same as it is moved along the flat surface of the table. Under linear conditions, the order in which $x$ and $y$ partial derivatives are taken does not matter, since the tangent space and the flat surface are the same. Once curvature is involved, the order in which $x$ and $y$ partial derivatives are taken matters, since the curved surface and the tangent plane are not the same [6]. That is why the vector did not change on the flat surface of the table in terms of direction, but did on the curved surface of the globe.
2.3 Black Holes

So what exactly are black holes and do they exist? Although black holes seem to be mysterious objects in the universe that were arguably made up by physicists, they do exist and are more common than people think. Consider flat spacetime. When a mass is added, there is curvature around that object. This curvature is caused by the distortion of spacetime. Thus, the greater the mass in spacetime, the greater the distortion. With a great enough mass, the distortion will create a black hole. The formation of a black hole occurs when the $r$ value of the distortion is equal to $2GM$ [3]. This distortion causes the two dimensional representation of Gravity of the black hole to be equal to zero.

The research presented in this thesis for converting mass in General Relativity to mass in Newton-Cartan Gravity, as well as the orbits of massless and massive particles, were computed for four types of black holes. These four black holes include Schwarzschild, Reissner-Nördstrom, Kerr, and Kerr-Newman. The Schwarzschild black hole is a stationary black hole with mass. The Reissner-Nördstrom black hole is a stationary black hole with both electrical charge and mass. The Kerr black hole is a rotating black hole with mass. The Kerr-Newman black hole is a rotating black hole with both electrical charge and mass [3].

Each of these black holes has their own $f(r)$ function. This function describes the Newtonian Gravitational Field of that particular black hole. The $f(r)$ is known for the Schwarzschild and Reissner-Nördstrom black holes and the $f(r)$ for the Kerr and Kerr-Newman black holes came from the research
conducted by Leo Rodriguez [8]. The metrics and metric tensors described above in the Metrics and Metric Tensors portion of the background section were calculated for each black hole. In addition, the Christoffel symbols for each black hole were calculated, due to their importance in the calculus form of the hypothesis.

As stated previously, General Relativity can be used to solve problems in binary solar systems, solar systems with two suns. If one of these stars becomes a black hole it is possible for that blackhole to draw heat from the other star. A black hole exhibiting this type of behavior is referred to as the X-ray binary black hole. The heat given off by this blackhole can be measured through the use of X-rays [3]. In fact, physicists have discovered a few X-ray binary black holes in the universe this way.

There is a duality that exists between thermodynamics and the laws of black hole mechanics [8]. This duality tells us that black holes act as thermodynamic bodies with entropy and temperature.

Holography can be shown with black holes through the gravitational metric and the $f(r)$ function. The two dimensional holographic representation of the gravitational field in four dimensions is given by $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2$. In this context, $ds^2$ is the equation of the gravitational metric for a black hole. Knowing the $f(r)$ function for these black holes allows us to make these calculations through the use of holographic scaling. This method is especially true for the Kerr and Kerr Newman black holes, since they have numbers other than zero off of the diagonal in the metric.

From the $f(r)$ function of each black hole, the $V(r)$ function, the potential
energy with respect to the distance from the center of the black hole can be determined.
2.4 Holography

The holographic principle was first introduced to the field of physics in 1993 by Gerard ’t Hooft \[1\]. Although this was the first introduction to the idea of holography in the field of physics, other fields had been using the concept for quite some time. In fact, some say that the concept of holography was foreshadowed by the Ancient Greek philosopher, Plato in his 'Allegory of the Cave' \[1\]. In this allegory, Plato discusses the lives of prisoners that only see shadows on the wall. To them, these shadows are reality, as they are unaware of the fact the these shadows are cast from the fire illuminating the real world objects behind them. Through this allegory, Plato is demonstrating that a three dimensional object can be projected onto a two dimensional surface, which fits the definition of holography. Similarly, as children we also demonstrate holography first hand when making shadow puppets with our hands using flashlights. The three dimensional configurations we twist our hands into project a two dimensional image on the wall when a flashlight is shined on them.

Holography can be defined as encoding an object of multiple dimensions on a surface of fewer dimensions. This concept can be visualized through an example using hamster balls. If a person tries to fit their entire body into a hamster ball, they will probably only be able to fit their hands inside. However, if they have a large enough hamster ball they will be able to fit their entire body inside. This example shows that the greater the radius of the sphere, the greater the amount of the desired object one can fit into the sphere. Applying this idea to physics, take a four dimensional mass($M_{ADM}$), and enclose it in a
two sphere. Therefore, the two sphere might not enclose the entire four dimensional mass. Thus, the radius of this two sphere can be extended to infinity so the entire mass will be enclosed.

Figure 4: The four dimensional $M_{ADM}$ is partially enclosed by a two sphere, but if the radius of that two sphere is extended to infinity, then the entire four dimensional mass can be enclosed.

Now, at radial infinity, there is holography since the entire four dimensional mass is enclosed in the two sphere.
2.5 Lagrangian and Action Integral

The Lagrangian \( L \) is the difference between kinetic energy \( K \) and potential energy \( V \), \( L = K - V \). The Euler Lagrange equations are satisfied at critical points where the action is unchanged. The action of a Lagrangian and the Euler Lagrange equations can be written as

\[
S = \int L(q, \dot{q}) \, dt
\]

and

\[
\frac{\partial L}{\partial q} - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) = 0,
\]

respectively, where \( q \) is the position of a particle in classical mechanics with respect to time, and \( \dot{q} \) is the derivative of \( q \). The action for Gravity is the integral that was exploited to get rid of two degrees of freedom, \( \theta \) and \( \phi \) to reduce a four dimensional of Gravity to a two dimensional theory coupled with a Dilaton using the holographic principle. The action integral for Gravity is

\[
S_{Gravity} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R,
\]

and the action integral for Conformal Weyl Gravity is

\[
S_{ConformalWeylGravity} = \alpha \int d^4x \sqrt{-g} W_{\mu \alpha \beta} W^{\mu \alpha \beta}.
\]

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2.6 Dilaton Gravity

Dilation Gravity is Gravity paired with a Dilaton field. Therefore, in order to understand what Dilaton Gravity is, we must first understand Dilaton fields. The Dilaton field is a dimensionless particle that comes into play in theories that have been compacted as to preserve the information from the original. A Dilaton field provides a quantization of Gravity, since it involves looking at Gravity at high energies. It also provides dynamics to scale and size. Connections between Dilaton fields and black holes allow physicists to understand the geometry and thermodynamics of black holes [6].

A Dilaton can also be referred to as a radion. A radion is a scalar, which represents the additional dimensions. This allows physicists to use Dilaton to break complex problems into an action potential, multiplied by a Dilaton. The research presented in this thesis takes four dimensional General Relativity and reduces it through the principle of holography to two dimensional Dilaton Gravity, a Dilaton times an action potential.
2.7 Weyl Tensor and Weyl Gravity

The Weyl tensor is similar to the Riemann tensor because it is a rank 4 tensor that measures curvature. However, the Weyl tensor does not change based on the scale of spacetime [3]. This means that 50% of spacetime and full spacetime would produce the same Weyl tensor. The equation for the Weyl tensor is written below.

\[ W_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \left( g_{\mu\beta}R_{\alpha\nu} + g_{\nu\alpha}R_{\beta\mu} - g_{\mu\alpha}R_{\beta\nu} - g_{\nu\beta}R_{\alpha\mu} \right) - \frac{2}{n-2} + \frac{(g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu})R}{(n-1)(n-2)} \]

where \( n \) is the number of dimensions.

The Weyl tensor is equal to the Riemann tensor plus combinations of the gravitational metric, Ricci tensor, and Ricci scalar. This combination of components is what makes the Weyl tensor independent of the scale of spacetime in its measurement of curvature. Weyl Gravity is given by the square of the Weyl tensor.
3 Results and Discussion

3.1 Mass in General Relativity to Newton Cartan

This first section of results demonstrates the conversion of mass in General Relativity to mass in Newton-Cartan Gravity. As mentioned in the introduction, Newton-Cartan Gravity is the low energy approximation of Einstein’s full theory of Gravity, which is General Relativity. The mass in General Relativity was converted to mass in Newton-Cartan Gravity through the holographic principle. It was then determined that if the radius of a two sphere is extended to infinity, it will be able to contain the entire four dimensional mass ($M_{ADM}$).

The black holes that this research focuses on are the Schwarzschild, Reissner-Nördstrom, Kerr, and Kerr-Newman. Using the $f(r)$, which describes the Newtonian gravitational field for each of these four black holes, the gravitational metric for each black hole was calculated. Through various calculations explained in detail in the Tensor and Tensor Calculus as well as the Metrics and Metric Tensors sections of the background portion of this thesis, the Christoffel symbol of each black holes is calculated, by following through with the application of the hypothesis extended in calculus based on the idea that there is holography at radial infinity. As a result, the integral simplified down to only $M$ for each of the black holes showing that the total mass ($M_{ADM}$) is equal to $M$ at radial infinity. Therefore, showing that holography can be used to show how a four dimensional mass can be enclosed entirely in a two sphere.

For each black hole, the metric, inverse metric, Christoffel connection, Riemann tensor, Ricci tensor, Ricci scalar, and $M_{ADM}$ were calculated in two
dimensions using the respective f(r) function for each blackhole.

**Schwarzschild Blackhole Calculations in 2D**

\[ f(r) = 1 - \frac{2GM}{r} \]

**Metric**

\[
\begin{pmatrix}
-1 + \frac{2GM}{r} & 0 \\
0 & \frac{1 - \frac{2GM}{r}}{r^2}
\end{pmatrix}
\]

**Inverse Metric**

\[
\begin{pmatrix}
\frac{r}{2GM-r} & 0 \\
0 & 1 - \frac{2GM}{r}
\end{pmatrix}
\]

**Christoffel Connection**

\[ \left( ((0, -\frac{GM}{2GM-r}), (-\frac{GM}{2GM-r}^2), 0)) , \right.

\[ ((-\frac{GM(2GM-r)}{r^3}, 0), (0, \frac{GM}{2GM-r^2})) \]

**Riemann Tensor**

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{2G(2GM-r)}{r^4} & -\frac{2GM(-2GM-r)}{r^4} & 0 \\
0 & \frac{2GM}{(2GM-r)^2} & -\frac{2G}{(2GM-r)^2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

**Ricci Tensor**

\[
\begin{pmatrix}
\frac{2GM(2GM-r)}{r^4} & 0 \\
0 & -\frac{2GM}{(2GM-r)^2}
\end{pmatrix}
\]

**Ricci Scalar**

\[ \frac{4GM}{r^4} \]
\[
M_{ADM} = \lim_{r \to \infty} \frac{1}{G} r^2 \Gamma_{11}^2
= \lim_{r \to \infty} \frac{1}{G} r^2 \left( \frac{-GM(2GM - r)}{r^3} \right)
= M
\]

Reissner-Nördstrom Blackhole Calculations in 2D

\[
f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}
\]

Metric:
\[
\begin{pmatrix}
-1 + \frac{2GM}{r} & -\frac{GQ^2}{r^2} & 0 \\
0 & \frac{r^2}{r(-2GM+r)+GQ^2} & 0 \\
0 & 0 & \frac{r^2}{r(-2GM+r)+GQ^2}
\end{pmatrix}
\]

Inverse Metric:
\[
\begin{pmatrix}
-\frac{r^2}{r(-2GM+r)+GQ^2} & 0 & 0 \\
0 & \frac{r^2}{r(-2GM+r)+GQ^2} & 0 \\
0 & 0 & \frac{r^2}{r(-2GM+r)+GQ^2}
\end{pmatrix}
\]

Christoffel Connection:
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{G(Mr-Q^2)}{r(r(-2GM+r)+GQ^2)} & \frac{G(Mr-Q^2)}{r(r(-2GM+r)+GQ^2)} \\
0 & \frac{G(Mr-Q^2)}{r(r(-2GM+r)+GQ^2)} & 0
\end{pmatrix}
\]

Riemann Tensor:
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{G(2Mr-3Q^2)((2GM-r)GQ^2)}{r^6} & \frac{G(2Mr-3Q^2)((2GM-r)GQ^2)}{r^6} \\
0 & \frac{2GMr+3GQ^2}{r^3(r(-2GM+r)+GQ^2)} & \frac{G(2Mr-3Q^2)((2GM-r)GQ^2)}{r^6(r(-2GM+r)+GQ^2)} \\
0 & 0 & 0
\end{pmatrix}
\]

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Ricci Tensor =
\[
\begin{pmatrix}
\frac{G(2Mr-3Q^2)((2GM-r-GQ^2)}{r^6} & 0 \\
0 & \frac{G(2Mr-3Q^2)}{r^4(r(-2GM+r)+GQ^2)}
\end{pmatrix}
\]

Ricci Scalar = \frac{4GMr-6GQ^2}{r^4}

\[M_{ADM} = \lim_{r \to \infty} \frac{1}{G} r^2 \Gamma^2_{11} \]
\[= \lim_{r \to \infty} \frac{1}{G} r^2 \left( \frac{G(Mr - Q^2)((2GM - r)r - GQ^2)}{r^5} \right) \]
\[= M\]

Kerr Blackhole Calculations in 2D

\[f(r) = \frac{1-2GMr+a^2}{r^2+a^2} \]

Metric = \[
\begin{pmatrix}
\frac{-1+a^2-2GMr}{a^2+r^2} & 0 \\
0 & \frac{a^2+r^2}{1+a^2-2GMr}
\end{pmatrix}
\]

Inverse Metric = \[
\begin{pmatrix}
\frac{-a^2+r^2}{1+a^2-2GMr} & 0 \\
0 & \frac{1+a^2-2GMr}{a^2+r^2}
\end{pmatrix}
\]

Christoffel Connection = \[
(((0, -\frac{a^2(2GM+r)+r(-1+GMr)}{(1+a^2-2GMr)(a^2+r^2)}), (-\frac{a^2(2GM+r)+r(-1+GMr)}{(1+a^2-2GMr)(a^2+r^2)}), 0))
\]
\[
((\frac{-1+a^2-2GMr)(r-GMr^2+a^2(2GM+r)}{(a^2+r^2)^2}, 0); (0, \frac{r-GMr^2+a^2(2GM+r)}{(1+a^2-2GMr)(a^2+r^2)})\))
\]

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Riemann Tensor =
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & (1+a^2-2GMr)(a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2)) & \frac{a^2-2GMr(a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2))}{(a^2+r^2)^4} & 0 \\
0 & \frac{a^2-2GMr(a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2))}{(1+a^2-2GMr)(a^2+r^2)^2} & (1+a^2-2GMr)(a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2)) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Ricci Tensor =
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2)}{(a^2+r^2)^4} & 0 & \frac{a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2)}{(1+a^2-2GMr)(a^2+r^2)^2} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Ricci Scalar =
\[
2\frac{(a^4+r^2(-3+2GMr)+a^2(1-6GMr-3r^2))}{(a^2+r^2)^3}
\]

\[
M_{ADM} = \lim_{r \to \infty} \frac{1}{G} r^2 \frac{\Gamma^2}{r^2 + a^2}
\]
\[
= \lim_{r \to \infty} \frac{1}{G} r^2 \frac{-(a^2 + r(-2M + r))(r(-Mr) + a^2 M (\cos \theta)^2)}{(r^2 + a^2 (\cos \theta)^2)^3}
\]
\[
= M
\]

Kerr Newman Calculations in 2D

\[
f(r) = \frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2} = \frac{\Delta}{r^2 + a^2}
\]

Metric =
\[
\begin{pmatrix}
-\frac{a^2+Q^2-2Mr+r^2}{a^2+r^2} & 0 \\
0 & \frac{a^2+r^2}{a^2+Q^2-2Mr+r^2}
\end{pmatrix}
\]

Inverse Metric =
\[
\begin{pmatrix}
-\frac{a^2+r^2}{a^2+Q^2-2Mr+r^2} & 0 \\
0 & \frac{a^2+Q^2-2Mr+r^2}{a^2+r^2}
\end{pmatrix}
\]

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Christoffel Connection = \(((0, -\frac{a^2 M + r(-Q^2 + Mr)}{(a^2 + r^2)(a^2 + Q^2 + r(-2M + r))}), (-\frac{a^2 M + r(-Q^2 + Mr)}{(a^2 + r^2)(a^2 + Q^2 + r(-2M + r))}, 0)),
\((-\frac{a^2 Q^2 + r(-2M + r))(a^2 M + r(Q^2 - Mr))}{(a^2 + r^2)^3}, 0), (0, -\frac{a^2 M + r(Q^2 - Mr)}{(a^2 + r^2)(a^2 + Q^2 + r(-2M + r)))}) \)

Riemann Tensor =
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{a^2 Q^2 + r(-2M + r))(-a^2(Q^2 - 6Mr) + r^2(3Q^2 - 2Mr))}{(a^2 + r^2)^4} & 0 \\
0 & -\frac{a^2(Q^2 - 6Mr) + r^2(3Q^2 - 2Mr)}{(a^2 + r^2)^2(a^2 + Q^2 + r(-2M + r))} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Ricci Tensor =
\[
\begin{pmatrix}
\frac{a^2 Q^2 + r(-2M + r))(-a^2(Q^2 - 6Mr) + r^2(3Q^2 - 2Mr))}{(a^2 + r^2)^4} & 0 \\
0 & \frac{a^2(Q^2 - 6Mr) + r^2(-3Q^2 + 2Mr)}{(a^2 + r^2)^2(a^2 + Q^2 + r(-2M + r))} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Ricci Scalar = \(2\frac{a^2(Q^2 - 6Mr) + r^2(-3Q^2 + 2Mr)}{(a^2 + r^2)^3}\)

\[M_{ADM} = \lim_{r \to \infty} \frac{1}{G} r^2 F_{11}^2 = \lim_{r \to \infty} r^2 (-\frac{a^2 + Q^2 + r(-2M + r))(a^2 M + r(Q^2 - Mr))}{(a^2 + r^2)^3}) = M \]
These results can be found in the table below, which highlights the $f(r)$ function, $\Gamma_{11}^2$, and $M_{ADM}$.

<table>
<thead>
<tr>
<th>Black Hole</th>
<th>$f(r)$</th>
<th>$\Gamma_{11}^r = \Gamma_{11}^2$</th>
<th>$M_{ADM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwarzschild</td>
<td>$1 - \frac{2GM}{r}$</td>
<td>$\frac{-GM(2GM-r)}{r^3}$</td>
<td>$M$</td>
</tr>
<tr>
<td>Reissner-Nördstrom</td>
<td>$1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$</td>
<td>$\frac{G(-Q^2+Mr)(r^2+G(Q^2-2Mr))}{(a^2+2r^2+2a^2\cos2\theta)^3}$</td>
<td>$M$</td>
</tr>
<tr>
<td>Kerr</td>
<td>$\frac{1-2GM+ar^2}{r^2+a^2}$</td>
<td>$\frac{4M(-a^2+(2M-r)r)(a^2-2r^2+a^2\cos2\theta)}{(a^2+2r^2+2a^2\cos2\theta)^3}$</td>
<td>$M$</td>
</tr>
<tr>
<td>Kerr-Newman</td>
<td>$\frac{\Delta}{r^2+a^2}$</td>
<td>$\frac{-(a^2+Q^2+r(-2M+r))(r(Q^2-Mr)+a^2M\cos\theta^2)}{(r^2+a^2\cos\theta^2)^3}$</td>
<td>$M$</td>
</tr>
</tbody>
</table>

Table 5: This table highlights the $f(r)$ function, $\Gamma_{22}^1$, and $M_{ADM}$ for the Schwarzschild, Reissner Nördstrom, Kerr, and Kerr Newman black holes.

Since each black hole has an $f(r)$ function, and the limit of each of these black holes was $M$, a generalized equation involving the $f(r)$ function was developed for finding metric, inverse metric, Christoffel connection, Riemann tensor, Ricci tensor, Ricci scalar, and the limit in two dimensions. The generalized equations computed for each of these are listed below.

Metric = \begin{pmatrix} -f[r] & 0 \\ 0 & \frac{1}{f[r]} \end{pmatrix}

Inverse Metric = \begin{pmatrix} -\frac{1}{f[r]} & 0 \\ 0 & f[r] \end{pmatrix}

Christoffel Connection = (\{(0, \frac{f'[r]}{2f[r]}), (\frac{f''[r]}{2f[r]}, 0)\}, (\frac{1}{2}f[r], f'[r], 0), (0, -\frac{f''[r]}{2f[r]}) \})
Riemann Tensor = \[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{1}{2} f[r] f''[r] & 0 & 0 & 0 \\
0 & \frac{f''[r]}{2f[r]} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Ricci Tensor = \[
\begin{pmatrix}
\frac{1}{2} f[r] f''[r] & 0 \\
0 & -\frac{f''[r]}{2f[r]}
\end{pmatrix}
\]

Ricci Scalar = -f''[r]

\[M_{ADM} = \lim_{r \to \infty} \frac{1}{8\pi} r^2 \Gamma^2_{11}\] for Schwarzschild and Reissner-Nördstrom black holes

\[M_{ADM} = \lim_{r \to \infty} r^2 \Gamma^2_{11}\] for Kerr and Kerr Newman black holes.

These equations are useful, as they allow for any black hole calculations to be computed once the f(r) function is known. This provides a general methodology for the complicated calculations of black holes.
3.2 Orbits

Aside from computing the mass of each of the four black holes, the orbits of both massless and massive particles for each were also derived. These orbits were derived using the concept of holography.

The effective potentials in General Relativity were found for the Schwarzschild, Reissner-Nördstrom, Kerr, and Kerr-Newman black holes through graphing the orbits of massless and massive particles for each of these four black holes. The $V(r)$ function is the potential of a particle with unit mass and energy. The types of particles that $V(r)$ is concerned with in this research are massless and massive particles. In general, these particles move until $V(r)$ is equal to the energy of the particular particle. This point is called the turning point after which the particle moves in the other direction [3]. Each graph has a $V(r)$ evaluated with an angular momentum from one to five. For the purposes of these graphs, $GM$ was set equal to zero to make the $V(r)$ functions unitless. $V(r) = \frac{f(r)}{2} \left( \frac{L^2}{r^2} + \epsilon \right)$, where $L$ is the angular momentum and $\epsilon$ is zero for massless particles and one for massive particles. In the graphs presented in this section, $x$ is equivalent to $r$, representing the distance between the center of the black hole and the particle. Therefore, $V(x)$ is equivalent to $V(r)$.

The figures in this section show the orbits of massive and massless particles in General Relativity for the Schwarzschild black hole. These graphs plot the potential energy on the $y$-axis and the distance from the center of the black hole on the $x$-axis.
Figure 5: The orbits for massless (top) and massive (bottom) particles were graphed for the Schwarzschild black hole with the potential energy $V(x)$ versus $x$, the radius in meters they are from the center of the black hole.
For the Schwarzschild black hole, the relative maximum of each curve has a slope equal to zero. At this point, the object orbiting the black hole is stable and can continue to orbit the black hole. However, any slight shift can cause the object to either be sucked into the black hole or sent out to infinity. This type of trend can be observed for all four types of black holes, in general.

For massless particles, the slope is equal to zero when $x$ is equal to three since a maximum occurs at this point. The greater the angular momentum, the steeper the peak. This relationship indicates that the greater the angular momentum, the more likely a tiny shift in the position of the massless object will cause the object to be sent out to infinity. Yet, a lower angular momentum provides the object with a better chance of staying in its orbit. For massive particles, a similar trend at $x$ equals three can be seen. The lower the angular momentum, the more likely a shift could send the object into the black hole.

For the Reissner-Nördstrom black hole, there is a relative minimum followed by a relative maximum for both graphs.
Figure 6: The orbits for massless (top) and massive (bottom) particles were graphed for the Reissner-Nördstrom black hole with the potential energy $V(x)$ versus $x$, the radius in meters they are from the center of the black hole.

This is difficult to see in the graph for massless particles, but a minimum occurs at $x$ equals one, and a maximum occurs at $x$ equals two. Thus, at $x$ equals two, a shift to the right would still send the object off to infinity. But
a shift to the left would not send the object into the black hole. Instead the object would stay around $x$ equals one, moving slightly out and then slightly in. The same rocking motion will occur with the massive particles in the minimum, as seen on the graph at $x$ equals one.

The Kerr black hole produces a parabola-like graph for both massless and massive particles.
Figure 7: The orbits for massless (top) and massive (bottom) particles were graphed for the Kerr black hole with the potential energy $V(x)$ versus $x$, the radius in meters they are from the center of the black hole.

Again, the trend where the object will oscillate out towards infinity then come back and have a shorter distance to the center of the black hole can be
seen.

The same goes for the Kerr-Newman black hole, but if the massless particle with a low angular momentum goes out, it will not necessarily come back and participate in the swinging motion. The massive particles will, however, participate in the swinging motion that has been described for the black holes above.
Figure 8: The orbits for massless (top) and massive (bottom) particles were graphed for the Kerr-Newman black hole with the potential energy $V(x)$ versus $x$, the radius in meters they are from the center of the black hole.

The orbits of massless and massive particles about the Schwarzschild, Reissner-Nörestrom, Kerr, and Kerr-Newman black holes exhibit similar trends. A slight shift in the position of either particle at a distance from the center of the black holes that is graphed as a relative maximum can either cause the particle to fly off into infinity or be sucked into the black hole. However,
when a relative minimum exists at a closer distance to the center of the black holes, the particle will exhibit the swinging motion, remaining around that orbit instead of being sucked into the black hole.
3.3 Dilaton Gravity

Through the concept of holography, a four dimensional vacuum solution for pure Gravity can be reduced to a two dimensional theory of Gravity. Thus, the two dimensional theory is a holographic representation of Gravity in four dimensions. The Ricci scalar is a pure theory of Gravity, and integrating it over $\phi$ and $\theta$ eliminates two degrees of freedom. As a result of simplifying, a Dilaton coupled with a two dimensional black hole, in terms of the $f(r)$ function remains.

$$ds^2 = \frac{e^{-2\Psi}}{\lambda}d\theta^2 + \frac{1}{f[r]}dr^2 - f[r]dt^2 + \frac{e^{-2\Psi[r]}}{\lambda}d\phi^2$$

\[
\text{Metric} = \begin{pmatrix}
-f[r] & 0 & 0 & 0 \\
0 & \frac{1}{f[r]} & 0 & 0 \\
0 & 0 & \frac{e^{-2\Psi[r]}}{\lambda^2} & 0 \\
0 & 0 & 0 & \frac{e^{-2\Psi[r](\sin\theta)^2}}{\lambda^2}
\end{pmatrix}
\]

\[
\text{Inverse Metric} = \begin{pmatrix}
-\frac{1}{f[r]} & 0 & 0 & 0 \\
0 & f[r] & 0 & 0 \\
0 & 0 & e^{2\Psi[r]\lambda^2} & 0 \\
0 & 0 & 0 & e^{2\Psi[r]\lambda^2(csc\theta)^2}
\end{pmatrix}
\]

\[
\text{Ricci Scalar (R)} = 2e^{2\Psi[r]\lambda^2} + f'[r]\Psi'[r] - f''[r] + f[r](-6\Psi'[r]^2 + 4\Psi''[r])
\]
Through the holographic principle, four dimensional General Relativity was reduced to a two dimensional theory of Gravity coupled with a Dilaton, Dilaton Gravity. Thus, the holographic principle reduces the complexity of problems. Although the two dimensional theory is easier to solve, it involves more components than the original four dimensional theory as to not lose any information in the reduction of dimensions.
3.4 Weyl Gravity

In addition, to using the holographic principle to go from four dimensional
General Relativity to two dimensional Dilaton Gravity, this principle was also
used to reduce Weyl Gravity from four to two dimensions.

The square of two Weyl tensors, also known as Weyl Gravity, was com-
puted. Weyl Gravity along with the square root of the negative determinant
of the gravitational metric are integrated with respect to $\phi$ and $\theta$ to eliminate
two degrees of freedom. As a result, what remains is a two dimensional theory
of Weyl Gravity coupled with a Dilaton.

$$ds^2 = -f[a]dt^2 + \frac{1}{f[a]}da^2 + e^{-2\Psi[a]}l^2d\theta^2 + e^{-2\Psi[a]}l^2(\sin \theta)^2d\phi^2$$

$$\text{Metric} = \begin{pmatrix}
- f[a] & 0 & 0 & 0 \\
0 & \frac{1}{f[a]} & 0 & 0 \\
0 & 0 & e^{-2\Psi[a]}l^2 & 0 \\
0 & 0 & 0 & e^{-2\Psi[a]}l^2(\sin \theta)^2
\end{pmatrix}$$

$$\text{Inverse Metric} = \begin{pmatrix}
- \frac{1}{f[a]} & 0 & 0 & 0 \\
0 & f[a] & 0 & 0 \\
0 & 0 & \frac{e^{2\Psi[a]}}{l^2} & 0 \\
0 & 0 & 0 & \frac{e^{2\Psi[a]}(\csc \theta)^2}{l^2}
\end{pmatrix}$$
\[ W^{\mu\nu\alpha\beta} = \{ \{ 0, 0, 0, 0 \}, \{ 0, 0, 0, 0 \}, \{ 0, 0, 0, 0 \}, \{ 0, 0, 0, 0 \} \}, \]
\[ \{ 0, \frac{1}{6} \left( -2e^{2\Psi[r]} + 2f'[r]\Psi'[r] + f''[r] + 2f[r]\Psi''[r] \right), 0, 0 \}, \]
\[ \{ \frac{1}{6} \left( -2e^{2\Psi[r]} - 2f'[r]\Psi'[r] - f''[r] - 2f[r]\Psi''[r] \right), 0, 0, 0, \{ 0, 0, 0, 0 \} \}, \]
\[ \{ 0, 0, \frac{1}{12} e^{2\Psi[r]} (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ \frac{1}{12} e^{2\Psi[r]} (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] + l^2 f''[r] + 2f^2[f[r]\Psi''[r]]), 0, 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{12} e^{2\Psi[r]} \csc[\theta]^2 (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{12} e^{2\Psi[r]} (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] + f''[r] - 2f^2[f[r]\Psi''[r]]), 0, 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{6} \left( -2e^{2\Psi[r]} + 2f'[r]\Psi'[r] + f''[r] - 2f[r]\Psi''[r] \right), 0, 0, 0 \}, \]
\[ \{ 0, 0, -\frac{1}{12} e^{2\Psi[r]} f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0 \}, \]
\[ \{ 0, \frac{1}{12} e^{2\Psi[r]} f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] + f''[r] - 2f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{12} e^{2\Psi[r]} \csc[\theta]^2 f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ 0, 0, \frac{1}{12} e^{2\Psi[r]} \csc[\theta]^2 f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] + f''[r] - 2f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{12} e^{2\Psi[r]} (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0 \}, \]
\[ \{ 0, 0, -\frac{1}{12} e^{2\Psi[r]} f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{12} e^{2\Psi[r]} \csc[\theta]^2 (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0 \}, \]
\[ \{ 0, 0, -\frac{1}{12} e^{2\Psi[r]} \csc[\theta]^2 f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0, 0 \}, \]
\[ \{ 0, 0, 0, \frac{1}{12} e^{2\Psi[r]} \csc[\theta]^2 f[r] (2e^{2\Psi[r]} - 2f^2[f[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[f[r]\Psi''[r]]), 0, 0 \}, \}
\]
$$\{0, -\frac{1}{12}\text{e}^{2\Psi[r]} \text{csc}[\theta]^2 f[r] (2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r]), 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\},$$

$$\{0, 0, -\frac{1}{6}\text{e}^{4\Psi[r]} \text{csc}[\theta]^2 (2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r])\},$$

$$\{0, 0, 0, 0, \frac{1}{6}\text{e}^{4\Psi[r]} \text{csc}[\theta]^2 (2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r]), 0\},$$

$$\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}.\}$$

$$W_{\mu\nu\alpha\beta} = \{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}.$$

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{0, 0, \frac{1}{6}e^{-4\Psi[r]}l^2 \sin[\theta]^2(2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r]), 0} \},
\{\{0, 0, 0, -\frac{1}{12}e^{-2\Psi[r]}f[r]\sin[\theta]^2(2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r])\}, \{0, 0, 0, 0\}, \frac{1}{12}e^{-2\Psi[r]}f[r]\sin[\theta]^2(2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r])\}, 0, 0, 0\},
\{0, 0, 0, 0, \frac{1}{12}e^{-2\Psi[r]}f[r]\sin[\theta]^2(2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r])\}, 0, 0, 0\},
\{0, 0, 0, 0, \frac{1}{12}e^{-2\Psi[r]}f[r]\sin[\theta]^2(2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r])\}, 0, 0, 0\},
\{0, 0, 0, 0, \frac{1}{6}e^{-4\Psi[r]}l^2 \sin[\theta]^2(2e^{2\Psi[r]} - 2l^2 f'[r]\Psi'[r] - l^2 f''[r] - 2l^2 f[r]\Psi''[r]), 0\},
\{\{0, 0, 0, 0, 0, 0, 0, 0\}\}}

\int_0^\pi d^2\theta W^{\mu\alpha\beta} W_{\mu\alpha\beta} \sqrt{-\det(g_{\mu\nu})} d\phi d\theta =
\frac{4e^{-2\Psi[a]}(-2e^{2\Psi[a]} + l^2 f'[a] \Psi'[a] + f''[a] + 2f[a] \Psi''[a])}{3l^2}
\left(\frac{16\pi l^2}{3} \int d^2x \sqrt{-\det(g_{\mu\nu})} e^{-2\Psi[r]} \left[-\frac{R}{2} + \nabla_{\mu} \nabla^{\mu} \Psi - \frac{e^{2\Psi[r]}}{l^2}\right]\right)^2

Similar to the results shown in the Dilaton Gravity results section, the holographic principle reduced Weyl Gravity from four dimensions to two dimensions. Again indicating that the holographic principle can be used to simplify complex calculations in physics, which in turn will help physicists to understand more theoretical components of the subject.
4 Conclusion

Through the holographic principle, complex problems in the field of physics can be broken down into simpler calculations. Although these calculations are easier to solve, they involve more components to them, which is why no information is lost when a four dimensional theory is reduced to a two dimensional one. This thesis has demonstrated that by extending the radius of a two sphere to infinity, an entire four dimensional mass can be enclosed and is equal to $M_{ADM}$. Both Dilaton Gravity and Conformal Gravity show how the holographic principle can reduce four dimensional pure Gravity into a two dimensional theory of Gravity coupled with a Dilaton.
References


