2015

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Recommended Citation

Gates, Jr., S. J.; Parker, J.; Rodgers, V. G. ; Rodriguez, L.; and Stiffler, K. (2015). An Extended Detailed Investigation of First and Second Order Supersymmetries for Off-Shell N=2 and N=4 Supermultiplets. *Symmetry* 7(2): 1080-1121. https://doi.org/10.3390/sym7021080

Symmetry **2015**, *7*, 1080-1121; doi:10.3390/sym7021080

OPEN ACCESS
symmetry
ISSN 2073-8994
www.mdpi.com/journal/symmetry

Article

An Extended Detailed Investigation of First and Second Order Supersymmetries for Off-Shell $\mathcal{N}$ = 2 and $\mathcal{N}$ = 4 Supermultiplets

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Academic Editor: Jonathan Bougie

Received: 5 May 2015 / Accepted: 11 June 2015 / Published: 16 June 2015

Abstract: This paper investigates the $d = 4$, $\mathcal{N} = 4$ Abelian, global Super-Yang Mills system (SUSY-YM). It is shown how the $\mathcal{N} = 2$ Fayet Hypermultiplet (FH) and $\mathcal{N} = 2$ vector multiplet (VM) are embedded within. The central charges and internal symmetries provide a plethora of information as to further symmetries of the Lagrangian. Several of these symmetries are calculated to second order. It is hoped that investigations such as these may yield avenues to help solve the auxiliary field closure problem for $d = 4$, $\mathcal{N} = 4$, SUSY-YM and the $d = 4$, $\mathcal{N} = 2$ Fayet-Hypermultiplet, without using an infinite number of auxiliary fields.

Keywords: supersymmetry; off-shell; ADS/CFT; Yang-Mills; extended supersymmetry; adinkra

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1. Introduction

The $\mathcal{N} = 4$ Super-Yang Mills (SUSY-YM) system is a very active area of study, and has become even more so over the past decade with the emergence of the AdS/CFT correspondence [1]. One very powerful aspect of this correspondence is that it relates a perturbation theory to a strongly coupled system. As $\mathcal{N} = 4$ SUSY-YM is a conformal field theory, an important undertaking has been to find dualities between string theory and theories that are more *QCD-like*. Klebanov and Strassler took a step in this direction in [2], where they unveiled a background which breaks the supersymmetry to $\mathcal{N} = 1$, while regulating the IR divergence behavior. Following this work, several other supersymmetry breaking backgrounds were discovered [3–6].

In parallel to the unveiling of these duality backgrounds, specific calculations were done showing duality to confining gauge theory calculations. Herzog and Klebanov showed duality in the tree level energy calculations between branes on the supergravity side and confining strings on the gauge theory side [7,8]. In this newly emerging gauge/gravity picture, Regge trajectories were resurrected from the old dual resonance models and reinvestigated by Pando Zayas, Sonnenschein, and Vaman in [9], including some one loop level calculations. Most recently, one loop corrections to the k-string energy have been investigated, the so-called Lüscher term. This emerges on the string theory side through the bosonic part of the D-brane energy, although in addition different one loop information of the *fermionic* part has also been unveiled [10–13]. A nice picture is developing which shows the dualities between objects on the string theory and gauge theory sides.

In this paper, we take a step back from this picture. Even though this is the best understood of the gauge/gravity dualities, the $d = 4$, $\mathcal{N} = 4$ SUSY-YM theory part of the correspondence itself still has unknown attributes. The most glaring issue is the auxiliary field closure problem: it is still unknown how to augment this theory with finite numbers of auxiliary fields such that the charges satisfy the following algebra:

$$\{\mathrm{D}_a^I, \mathrm{D}_b^J\} = 2\, i\, \delta^{IJ} (\gamma^\mu)_{ab} \partial_\mu \tag{1}$$

This is a problem which has been well known for at least thirty years. In 1981, Siegel and Rocek (SR) investigated a solution within the known framework that existed at the time and found a no-go theorem [14]. This result has been interpreted as the definitive statement on this issue.

However, there are some loose ends that challenge this conventional wisdom about the SR no-go theorem. The first of these is contained within the SR work itself. In an often overlooked final commentary in the work, the authors state a possible way to avoid the SR no-go theorem. It is also often overlooked that the derivation of the SR no-go theorem is based on a particular assumption of dynamics. In particular, the authors assume the gauge field is subject to the dynamics of the usual Yang-Mills action. It is simple to consider a different starting point. It is easy to negate this assumption.

Though mostly unknown, the action for the ABJM model [15] together with a discussion of 3D, $\mathcal{N} = 6$ superconformal invariance first appeared in works written in the period of 1991–1995 on the importance of Chern-Simons models [16–19]. So instead of considering the fields of a vector multiplet in 4D that realizes $\mathcal{N} = 4$ SUSY or a hypermultiplet in 4D that realizes $\mathcal{N} = 2$ SUSY, one could attempt to construct respective 3D Chern-Simons models with $\mathcal{N} = 8$ SUSY or $\mathcal{N} = 4$ SUSY that are based on the dimensional reduction of 4D multiplets. The SR no-go theorem cannot be applied to such constructions! Thus, the study of 3D Chern-Simons theories provides a new way to attack this very old problem.

The methods in harmonic [20,21] or projective [22,23] superspace absolutely offer solutions, however these add an infinite number of auxiliary fields. In this paper we offer an in-depth analysis of the Lagrangian symmetries generated by the central charges and internal symmetries of the algebra as a possible window into algebraic closure with a finite number of auxiliary fields. To the knowledge of the authors, these symmetries have never been discussed in this detail; almost certainly not in the 4-D Majorana component notation that is used in this paper. In short, we are trying to push the bounds of knowledge further to understand how the algebra fails to close with a finite number of auxiliary fields. Furthermore, this paper analyzes the central charges and internal symmetries, or lack thereof, of other SUSY systems embedded into the overarching $d = 4\,\mathcal{N} = 4$ SUSY-YM system.

This paper is structured as follows. We begin by showing how the Abelian $d = 4$, $\mathcal{N} = 4$ super Yang-Mills (SUSY-YM) system can be made to split into the $\mathcal{N} = 2$ vector multiplet (VM), which closes, and the $\mathcal{N} = 2$ Fayet Hypermultiplet (FH) systems, which does not [24]. Then we show the main result: the recovery of many first and second order supersymmetries from the central charges and internal symmetries of this algebra.

Unless otherwise specified throughout the document, our notation convention is as follows. Capital Latin indices are euclidean and go from one to three: $I, J, K, \ldots = 1, 2, 3$. Lower case Latin indices $i, j, k, m, \cdots = 1, 2$ are also Euclidean. This is not to be confused with the spinor indices, which are the other half of the lower case latin alphabet $a, b, c, d, \cdots = 1, 2, 3, 4$, ranging from one to four. Greek indices are four dimensional Minkowski space-time indices and go from zero to three: $\mu, \nu, \alpha, \beta, \cdots = 0, 1, 2, 3$. Symmetrization and antisymmetrization are defined without normalization:

$$\Lambda_{(\mu\nu)} = \Lambda_{\mu\nu} + \Lambda_{\nu\mu} \tag{2}$$

$$\Lambda_{[\mu\nu]} = \Lambda_{\mu\nu} - \Lambda_{\nu\mu} \tag{3}$$

2. Materials and Methods

This section presents the algebra for d = 4. $\mathcal{N} = 4$ SUSY-YM is laid out in component notation. The Lagrangian is presented, which is globally invariant to these transformations. Next, the algebra is uncovered, which of course does not close. It is shown how this algebra splits into both the $\mathcal{N} = 2$ FH and $\mathcal{N} = 2$ VM multiplets; the latter closes while the former does not. It is commented on how after reduction to the FH system, certain central charges and internal symmetries are removed from the algebra. Of course, all central charges and internal symmetries are removed from the algebra under reduction to the $\mathcal{N} = 2$ VM multiplet. The central charges and internal symmetries present in the algebra for SUSY-YM and FH unveil Lagrangian symmetries. These symmetries and the method with which they are unveiled is the main result of the paper, and they are catalogued in Section 3. The calculations to find the transformation laws and algebra were performed with *Mathematica*, along with calculations by hand to check the *Mathematica* code. The calculations to find the symmetries of the Lagrangian from the algebra and transformation laws were performed by hand.

2.1. $\mathcal{N} = 4$ *Transformation Laws*

The Lagrangian for the Abelian $d = 4$, $\mathcal{N} = 4$ SUSY-YM system

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}(\partial_\mu A^J)(\partial^\mu A^J) - \frac{1}{2}(\partial_\mu B^J)(\partial^\mu B^J) \\ & + i\frac{1}{2}(\gamma^\mu)^{ab}\psi_a^J\partial_\mu\psi_b^J + \frac{1}{2}(F^J)^2 + \frac{1}{2}(G^J)^2 \\ & - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}i(\gamma^\mu)^{cd}\lambda_c\partial_\mu\lambda_d + \frac{1}{2}\mathrm{d}^2\end{aligned} \tag{4}$$

is invariant with respect to the global supersymmetric transformations

$$\begin{aligned}\mathrm{D}_a A^J &= \psi_a^J \quad , \\ \mathrm{D}_a B^J &= i\,(\gamma^5)_a{}^b\,\psi_b^J \quad , \\ \mathrm{D}_a \psi_b^J &= i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A^J \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B^J \\ & \quad -\; i\,C_{a\,b}\,F^J \;+\; (\gamma^5)_{a\,b}G^J \quad , \\ \mathrm{D}_a F^J &= (\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b^J \quad , \\ \mathrm{D}_a G^J &= i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b^J \quad .\end{aligned} \tag{5}$$

$$\begin{aligned}\mathrm{D}_a\,A_\mu &= (\gamma_\mu)_a{}^b\,\lambda_b \quad , \\ \mathrm{D}_a\lambda_b &= -\;\frac{1}{2}(\sigma^{\mu\nu})_{ab}\,F_{\mu\nu} \;+\; (\gamma^5)_{a\,b}\,\mathrm{d} \quad , \\ \mathrm{D}_a\,\mathrm{d} &= i\,(\gamma^5\gamma^\mu)_a{}^b\;\partial_\mu\lambda_b \quad .\end{aligned} \tag{6}$$

$$\begin{aligned}\mathrm{D}_a^I A^J &= \delta^{IJ}\,\lambda_a - \epsilon^{IJ}{}_K\psi_a^K \quad , \\ \mathrm{D}_a^I B^J &= i\,(\gamma^5)_a{}^b\,[\,\delta^{IJ}\,\lambda_b + \epsilon^{IJ}{}_K\psi_b^K\,] \quad , \\ \mathrm{D}_a^I\psi_b^J &= \delta^{IJ}[\;\frac{1}{2}(\sigma^{\mu\nu})_{ab}\,F_{\mu\nu} \;+\; (\gamma^5)_{a\,b}\,\mathrm{d}\,] \\ & \quad -\;\epsilon^{IJ}{}_K[\;-i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A^K \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B^K \\ & \quad +\; i\,C_{a\,b}\,F^K \;+\; (\gamma^5)_{a\,b}G^K\;] \quad , \\ \mathrm{D}_a^I F^J &= (\gamma^\mu)_a{}^b\,\partial_\mu\,[\,\delta^{IJ}\,\lambda_b - \epsilon^{IJ}{}_K\psi_b^K\,] \quad , \\ \mathrm{D}_a^I G^J &= i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\,[\;-\delta^{IJ}\,\lambda_b + \epsilon^{IJ}{}_K\psi_b^K\,] \quad .\end{aligned} \tag{7}$$

$$\begin{aligned}\mathrm{D}_a^I\,A_\mu &= -\,(\gamma_\mu)_a{}^b\,\psi_b^I \quad , \\ \mathrm{D}_a^I\lambda_b &= i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A^I \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B^I \\ & \quad -\; i\,C_{a\,b}\,F^I \;-\; (\gamma^5)_{a\,b}G^I \quad , \\ \mathrm{D}_a^I\,\mathrm{d} &= i\,(\gamma^5\gamma^\mu)_a{}^b\;\partial_\mu\psi_b^I \quad ,\end{aligned} \tag{8}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9}$$

Our conventions for the gamma matrices are as in Appendix of [25]. Note here that D_a and D_a^I with $I = 1, 2, 3$ comprise a se of $\mathcal{N} = 4$ transformation laws.

These transformations are known as *zeroth* order symmetries of the Lagrangian. The main result of this paper will be the first and second order symmetries of the Lagrangian, and how they can be recovered from the algebra. We wish to return to the calculation of third order and higher symmetries in the future.

2.2. $\mathcal{N} = 4$ Algebra

In this section, we will discover the central charges and internal symmetries of this algebra which will lead us to the Lagrangian symmetries in Section 3. Using the shorthand

$$\chi = (A^I, B^I, F^I, G^I, \mathrm{d}, \psi_c^J, \lambda_c), \tag{10}$$

the algebra can be written

$$\{\mathrm{D}_a, \mathrm{D}_b\}\chi = 2i(\gamma^\mu)_{ab}\partial_\mu\chi, \quad \{\mathrm{D}_a, \mathrm{D}_b\}A_\nu = 2i(\gamma^\mu)_{ab}F_{\mu\nu} \tag{11}$$

and

$$\begin{aligned}
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}A^K =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu A^K - 2\epsilon^{IJK}(\gamma^5)_{ab}\mathrm{d}+\\
& - 2Z^{IJKM}[iC_{ab}F^M + (\gamma^5)_{ab}G^M],\\
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}B^K =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu B^K + 2\, i\epsilon^{IJK}C_{ab}\mathrm{d},\\
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}F^K =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu F^K + 2\epsilon^{IJK}(\gamma^5\gamma^\mu)_{ab}\partial_\mu\mathrm{d}+\\
& + 2Z^{IJKM}[-iC_{ab}\Box A^M + (\gamma^5\gamma^\mu)_{ab}\partial_\mu G^M]\\
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}G^K =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu G^K - 2\epsilon^{IJK}(\gamma^5\gamma^\mu)_{ab}\partial^\nu F_{\mu\nu}+\\
& - 2Z^{IJKM}[(\gamma^5)_{ab}\Box A^M + (\gamma^5\gamma^\mu)_{ab}\partial_\mu F^M]
\end{aligned} \tag{12}$$

$$\begin{aligned}
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}\mathrm{d} =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu\mathrm{d}+\\
& + 2\epsilon^{IJK}((\gamma^5)_{ab}\Box A^K - iC_{ab}\Box B^K + (\gamma^5\gamma^\mu)_{ab}\partial_\mu F^K)\\
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}A_\nu =& 2i\delta^{IJ}(\gamma^\mu)_{ab}F_{\mu\nu}+\\
& + 2\epsilon^{IJK}(iC_{ab}\partial_\nu A^K + (\gamma^5)_{ab}\partial_\nu B^K - (\gamma^5\gamma_\nu)_{ab}G^K)\\
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}\lambda_c =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu\lambda_c + i\epsilon^{IJK}[-C_{ab}(\gamma^\mu)_c{}^d + (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d+\\
& + (\gamma^5\gamma^\nu)_{ab}(\gamma^5\gamma_\nu\gamma^\mu)_c{}^d]\partial_\mu\psi_d^K\\
\{\mathrm{D}_a^I, \mathrm{D}_b^J\}\psi_c^K =& 2i\delta^{IJ}(\gamma^\mu)_{ab}\partial_\mu\psi_c^K - i\epsilon^{IJK}[-C_{ab}(\gamma^\mu)_c{}^d + (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d+\\
& + (\gamma^5\gamma^\nu)_{ab}(\gamma^5\gamma_\nu\gamma^\mu)_c{}^d]\partial_\mu\lambda_d+\\
& - iZ^{IJKM}[C_{ab}(\gamma^\mu)_c{}^d + (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d+\\
& + (\gamma^5\gamma^\nu)_{ab}(\gamma^5\gamma_\nu\gamma^\mu)_c{}^d]\partial_\mu\psi_d^M
\end{aligned} \tag{13}$$

and for the cross terms

$$\begin{aligned}
\{\mathrm{D}_a, \mathrm{D}_b^I\} A^J =& 2i\epsilon^{IJK} C_{ab} F^K \\
\{\mathrm{D}_a, \mathrm{D}_b^I\} B^J =& 2i\epsilon^{IJK} C_{ab} G^K \\
\{\mathrm{D}_a, \mathrm{D}_b^I\} F^J =& 2i\epsilon^{IJK} C_{ab} \Box A^K \\
\{\mathrm{D}_a, \mathrm{D}_b^I\} G^J =& 2i\epsilon^{IJK} C_{ab} \Box B^K \\
\{\mathrm{D}_a, \mathrm{D}_b^I\} \lambda_c =& 0
\end{aligned} \tag{14}$$

$$\begin{aligned}
\{\mathrm{D}_a, \mathrm{D}_b^I\} \mathrm{d} =& 0 \\
\{\mathrm{D}_a, \mathrm{D}_b^I\} A_\nu =& 2iC_{ab}\partial_\nu A^I - 2(\gamma^5)_{ab}\partial_\nu B^I \\
\{\mathrm{D}_a, \mathrm{D}_b^I\} \psi_c^J =& 2i\epsilon^{IJK} C_{ab} (\gamma^\mu)_c{}^d \partial_\mu \psi_d^K
\end{aligned} \tag{15}$$

where

$$Z^{IJKM} \equiv \delta^{IM}\delta^{JK} - \delta^{IK}\delta^{JM} \tag{16}$$

2.2.1. Central Charges and Internal Symmetries

We will use the notation (A^J, F^K) to indicate, for instance, the presence of a non-zero term involving the field F^K on the right hand side of the anti-commutator $\{\mathrm{D}_a^I, \mathrm{D}_b^J\}A^K$ and vice-versa. In this notation, we list the following fields which are coupled through a central charge or internal symmetry:

$$\left.\begin{aligned}
&(A^J, F^K), \quad (A^J, G^K), \quad (B^J, G^K), \\
&(A^J, \mathrm{d}), \quad (B^J, \mathrm{d}), \quad (G^J, A_\mu), \\
&(F^J, G^K), \quad (F^J, \mathrm{d}), \\
&(\psi_a^J, \lambda_b), \quad (\psi_a^J, \psi_b^K)
\end{aligned}\right\} \begin{array}{c}\text{fields coupled by a central charge}\\ \text{or internal symmetry}\end{array} \tag{17}$$

In addition, the algebra couples the following fields through a $U(1)$ gauge symmetry

$$(A_\mu, A^K), \quad (A_\mu, B^K), \quad \text{fields coupled through a gauge symmetry} \tag{18}$$

In Section 3, we will show how these central charges and internal symmetries can be used to uncover several first and second order Lagrangian symmetries. We note that this algebra is absent of central charges and internal symmetries between

$$\left.\begin{aligned}
&(F^J, A_\mu), \quad (A_\mu, \mathrm{d}), \quad (B^J, F^K), \\
&(B^J, A^K), \quad (G^J, \mathrm{d})
\end{aligned}\right\} \begin{array}{c}\text{fields } \textit{not} \text{ coupled through}\\ \text{a central charge}\\ \text{or internal symmetry}\end{array} \tag{19}$$

2.3. Reduction to $\mathcal{N} = 2$ Systems

Before we fully investigate the first and second order Lagrangian symmetries, we will investigate how to split the $\mathcal{N} = 4$ system into the $\mathcal{N} = 2$ FH and VM systems. When we do this, some of the central charges and internal symmetries vanish. In fact, in the case of the $\mathcal{N} = 2$ VM system *all* of these vanish, and the algebra has *no* information on first and second order Lagrangian symmetries. This is of course because the $\mathcal{N} = 2$ VM algebra closes. It is important to note that for reduction, we are considering only one pair of the six possible pairs of D-transformations. We leave the consideration of the other five pairs to future research.

Dropping the D^2_a and D^3_a transformations and making the following definitions:

$$\tilde{\mathrm{D}}^1_a \equiv \mathrm{D}_a, \quad \tilde{\mathrm{D}}^2_a \equiv \mathrm{D}^1_a \tag{20}$$

where $i = 1, 2$ labels the two supersymmetries of the embedded systems, we next make field redefinitions to manifest the embedded systems. The embedded $\mathcal{N} = 2$ VM system is composed of half of the fields of the $\mathcal{N} = 4$ system:

$$\begin{aligned} &A \equiv A^1, \quad B \equiv B^1, \quad F \equiv F^1, \quad G \equiv G^1, \\ &A_\mu, \quad \mathrm{d}, \quad \zeta^1_a \equiv \psi^1_a, \quad \zeta^2_a \equiv \lambda_a \end{aligned} \tag{21}$$

and the embedded $\mathcal{N} = 2$ FH system is composed of the other half

$$\begin{aligned} &\tilde{A}^1 \equiv A^2, \quad \tilde{A}^2 \equiv A^3, \quad \tilde{B}^1 \equiv B^2, \quad \tilde{B}^2 \equiv B^3, \\ &\tilde{F}^1 \equiv F^2, \quad \tilde{F}^2 \equiv F^3, \quad \tilde{G}^1 \equiv G^2, \quad \tilde{G}^2 \equiv G^3, \\ &\tilde{\psi}^1_a \equiv \psi^2_a, \quad \tilde{\psi}^2_a \equiv \psi^3_a \end{aligned} \tag{22}$$

2.3.1. Reduction to $\mathcal{N} = 2$ VM

The resulting $\mathcal{N} = 2$ VM algebra is

$$\begin{aligned} \tilde{\mathrm{D}}^i_a A &= \zeta^i_a, \\ \tilde{\mathrm{D}}^i_a B &= i(\gamma^5)_a{}^b \zeta^i_b, \\ \tilde{\mathrm{D}}^i_a F &= (\gamma^\mu)_a{}^b \partial_\mu \zeta^i_b, \\ \tilde{\mathrm{D}}^i_a G &= i(\sigma^3)^{ij}(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \zeta^j_b, \\ \tilde{\mathrm{D}}^i_a A_\mu &= i(\sigma^2)^{ij}(\gamma_\mu)_a{}^b \zeta^j_b, \\ \tilde{\mathrm{D}}^i_a \mathrm{d} &= i(\sigma^1)^{ij}(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \zeta^j_b, \\ \tilde{\mathrm{D}}^i_a \zeta^j_b &= \delta^{ij}(i(\gamma^\mu)_{ab}\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\partial_\mu B - iC_{ab}F) + (\sigma^3)^{ij}(\gamma^5)_{ab}G + \\ &\quad - i(\sigma^2)^{ij}\frac{1}{2}(\sigma^{\mu\nu})_{ab}F_{\mu\nu} + (\sigma^1)^{ij}(\gamma^5)_{ab}\mathrm{d}, \end{aligned} \tag{23}$$

where

$$(\sigma^1)^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2)^{ij} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\sigma^3)^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{24}$$

and

$$\zeta^1_b = \psi_b, \quad \zeta^2_b = \lambda_b. \tag{25}$$

The algebra reduces to

$$\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}\mathcal{V} = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\mathcal{V} \tag{26}$$

$$\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}A_\nu = 2i\delta^{ij}(\gamma^\mu)_{ab}F_{\mu\nu} + i(\sigma^2)^{ij}(2iC_{ab}\partial_\nu A - 2(\gamma^5)_{ab}\partial_\nu B). \tag{27}$$

where

$$\mathcal{V} = (A, B, F, G, \mathrm{d}, \psi_c, \lambda_c). \tag{28}$$

So this algebra closes up to gauge transformations and all the central charges and internal symmetries from the overarching $\mathcal{N} = 4$ algebra have vanished, aside from the $U(1)$ gauge symmetries. The algebra, therefore, contains no information on extra symmetries of the Lagrangian.

2.3.2. Reduction to $\mathcal{N} = 2$ FH

The transformation laws for the embedded $\mathcal{N} = 2$ FH system are

$$\begin{aligned}
\tilde{\mathrm{D}}^i_a\tilde{A}^j &= \delta^{ij}\tilde{\psi}^1_a + i(\sigma^2)^{ij}\tilde{\psi}^2_a,\\
\tilde{\mathrm{D}}^i_a\tilde{B}^j &= i(\gamma^5)_a{}^b\,[\,(\sigma^3)^{ij}\tilde{\psi}^1_b + (\sigma^1)^{ij}\tilde{\psi}^2_b\,],\\
\tilde{\mathrm{D}}^i_a\tilde{F}^j &= (\gamma^\mu)_a{}^b\partial_\mu[\,\delta^{ij}\tilde{\psi}^1_b + i(\sigma^2)^{ij}\tilde{\psi}^2_b\,],\\
\tilde{\mathrm{D}}^i_a\tilde{G}^j &= i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu[\,(\sigma^3)^{ij}\tilde{\psi}^1_b + (\sigma^1)^{ij}\tilde{\psi}^2_b\,],\\
\tilde{\mathrm{D}}^i_a\tilde{\psi}^1_b &= i(\gamma^\mu)_{ab}\partial_\mu\tilde{A}^i - iC_{ab}\tilde{F}^i + (\sigma^3)^{ij}[(\gamma^5)_{ab}\tilde{G}^j - (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{B}^j],\\
\tilde{\mathrm{D}}^i_a\tilde{\psi}^2_b &= (\sigma^2)^{ij}[-(\gamma^\mu)_{ab}\partial_\mu\tilde{A}^j + C_{ab}\tilde{F}^j]+\\
&\quad + (\sigma^1)^{ij}[(\gamma^5)_{ab}\tilde{G}^j - (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{B}^j]
\end{aligned} \tag{29}$$

with algebra

$$\begin{aligned}
\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}\tilde{A}^k &= 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{A}^k - 2i\tilde{Z}^{ijkm}C_{ab}\tilde{F}^m,\\
\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}\tilde{B}^k &= 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{B}^k - 2i\tilde{Z}^{ijkm}C_{ab}\tilde{G}^m,\\
\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}\tilde{F}^k &= 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k - 2i\tilde{Z}^{ijkm}C_{ab}\Box\tilde{A}^m,\\
\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}\tilde{G}^k &= 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k - 2i\tilde{Z}^{ijkm}C_{ab}\Box\tilde{B}^m,\\
\{\tilde{\mathrm{D}}^i_a, \tilde{\mathrm{D}}^j_b\}\tilde{\psi}^k_c &= 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{\psi}^1_c - 2i\tilde{Z}^{ijkm}C_{ab}(\gamma^\mu)_c{}^d\partial_\mu\tilde{\psi}^m_d
\end{aligned} \tag{30}$$

where

$$\tilde{Z}^{ijkm} \equiv \delta^{im}\delta^{jk} - \delta^{ik}\delta^{jm}, \quad i, j, k, m = 1, 2. \tag{31}$$

So only the couplings (A^J, G^K) and (F^J, G^K) have vanished from the overarching $\mathcal{N} = 4$ theory. Couplings still remain between $(\tilde{A}^j, \tilde{F}^k)$ and $(\tilde{B}^j, \tilde{G}^k)$ and $(\tilde{\psi}^i_a, \tilde{\psi}^j_b)$.

2.4. Uncovering First and Second Order Lagrangian Symmetries

This section shows the method with which we unveil first and second order Lagrangian symmetries. We show examples of how the procedure with more than one calculation will uncover the same symmetry. The full list of unique symmetries is unveiled in Section 3. The full list of calculations, including those unveiling redundant symmetries, are shown in Appendix. All such calculations were performed by hand.

2.4.1. First Order Bosonic Symmetries

Contracting the coupling from the anticommutator on A^J and F^J in Equation (14) with the Grassmann spinors ε^a and χ_I^b results in the first order bosonic symmetry of the Lagrangian

$$\delta^{(1)}_{BS3a}\begin{pmatrix} A^J \\ F^J \end{pmatrix} \equiv \frac{\varepsilon^a \chi_I^b}{2i}\{\mathrm{D}_a, \mathrm{D}_b^I\}\begin{pmatrix} A^J \\ F^J \end{pmatrix} = \varepsilon^a \chi_I^b \epsilon^{IJK} C_{ab}\begin{pmatrix} F^K \\ \Box A^K \end{pmatrix}. \tag{32}$$

Interestingly, contracting the coupling from the anticommutators on A^K and F^K in Equation (12) with the Grassmann spinors ε_I^a and χ_J^b results in a very similar first order bosonic symmetry of the Lagrangian

$$\delta^{(1)}_{BS3b}\begin{pmatrix} A^K \\ F^K \end{pmatrix} \equiv \varepsilon_I^a \chi_J^b Z^{IJKM} C_{ab}\begin{pmatrix} F^M \\ \Box A^M \end{pmatrix}. \tag{33}$$

In fact, these two symmetries are identical, and we can define them succinctly as:

$$\delta^{(1)}_{BS3}(T)\begin{pmatrix} A^K \\ F^K \end{pmatrix} \equiv T^{KM}\begin{pmatrix} F^M \\ \Box A^M \end{pmatrix}. \tag{34}$$

where

$$T^{KM} \equiv \begin{cases} \varepsilon_I^a \chi_J^b Z^{IJKM} C_{ab} \\ \text{or} \\ \varepsilon^a \chi_J^b \epsilon^{JKM} C_{ab} \end{cases} \tag{35}$$

This redundancy in the definition of T^{KM} begs the question: could a notation that somehow combines the underlying $\mathcal{N}=1$ vector multiplet and three copies of $\mathcal{N}=1$ chiral multiplets that comprise the $\mathcal{N}=4$ SUSY-YM multiplet result in a simplified definition of T^{KM}? At present, it is unknown how to do this and we wish to revisit this question in the future. Furthermore, we will see that redundancies are present in our calculations of other symmetries which leads us to define the variables P^K, Q^K, T^{KM}, $(U^\mu)^K$, W^{KM}, and $(V^\mu)^{KM}$ as:

$$\begin{gathered} P^K \equiv \varepsilon_I^a \chi_J^b \epsilon^{IJK}(\gamma^5)_{ab} \quad Q^K \equiv \varepsilon_I^a \chi_J^b \epsilon^{IJK} C_{ab}, \\ T^{KM} \equiv \begin{cases} \varepsilon_I^a \chi_J^b Z^{IJKM} C_{ab} \\ \text{or} \\ \varepsilon^a \chi_J^b \epsilon^{JKM} C_{ab} \end{cases}, \quad (U^\mu)^K \equiv \varepsilon_I^a \chi_J^b \epsilon^{IJK}(\gamma^5\gamma^\mu)_{ab}, \\ W^{KM} \equiv \begin{cases} \varepsilon^a \chi_J^b \epsilon^{JKM}(\gamma^5)_{ab} \\ \text{or} \\ \varepsilon_I^a \chi_J^b Z^{IJKM}(\gamma^5)_{ab} \end{cases}, \quad (V^\mu)^{KM} \equiv \begin{cases} \varepsilon^a \chi_J^b \epsilon^{JKM}(\gamma^5\gamma^\mu)_{ab} \\ \text{or} \\ \varepsilon_I^a \chi_J^b Z^{IJKM}(\gamma^5\gamma^\mu)_{ab} \end{cases} \end{gathered} \tag{36}$$

2.4.2. Second Order Bosonic Symmetries

By taking the commutators of each of the first order bosonic symmetries with each other, we reveal second order bosonic symmetries. This procedure will sometimes lead to redundant symmetries as in

$$\begin{aligned} \delta^{(2)}_{BS1a}(P_1,P_2)A^K &\equiv [\delta^{(1)}_{BS1}(P_1), \delta^{(1)}_{BS1}(P_2)]A^K = \Lambda^{KJ}_{1,1}(P_1,P_2)\Box A^J \\ \delta^{(2)}_{BS1b}(T_1,T_2)A^K &\equiv [\delta^{(1)}_{BS3}(T_1), \delta^{(1)}_{BS3}(T_2)]A^K = \Lambda^{JK}_{3,3}(T_1,T_2)\Box A^J \\ \delta^{(2)}_{BS1c}(W_1,W_2)A^J &\equiv [\delta^{(1)}_{BS5}(W_1), \delta^{(1)}_{BS5}(W_2)]A^J = \Lambda^{IJ}_{5,5}(W_1,W_2)\Box A^I \end{aligned} \tag{37}$$

where

$$\Lambda_{1,1}^{KJ}(P_1, P_2) \equiv P_{[1}^{K} P_{2]}^{J},$$
$$\Lambda_{3,3}^{KJ}(T_1, T_2) \equiv T_{[1}^{KM} T_{2]}^{MJ},$$
$$\Lambda_{5,5}^{IJ}(W_1, W_2) \equiv W_{[1}^{KI} W_{2]}^{JK} \qquad (38)$$

We can succinctly write these three redundant symmetries as one

$$\delta_{BS1}^{(2)}(\Lambda_1) A^K \equiv \Lambda_1^{[KJ]} \Box A^J \qquad (39)$$

where $(\Lambda_1)^{KJ}$ is an arbitrary 3×3 matrix and [ ] denotes antisymmetrization:

$$(\Lambda_1)^{[KJ]} = (\Lambda_1)^{KJ} - (\Lambda_1)^{JK}. \qquad (40)$$

2.4.3. First Order Fermionic Symmetries

Analogous to how we found the second order bosonic symmetries, we can uncover first order *fermionic* symmetries through calculations such as:

$$\delta_{FS19}^{(1)}(P) \begin{pmatrix} \mathrm{d} \\ \psi_b^J \end{pmatrix} \equiv -\varepsilon^a [\mathrm{D}_a, \delta_{BS1}^{(1)}(P)] \begin{pmatrix} \mathrm{d} \\ \psi_b^J \end{pmatrix} = \varepsilon^a P^J \begin{pmatrix} \Box \psi_a^J \\ i(\gamma^\mu)_{ab} \partial_\mu \mathrm{d} \end{pmatrix} \qquad (41)$$

All such possible calculations are listed in the Appendix A2, some of which are redundant as in the second order bosonic case.

3. Results

We list the first order bosonic symmetries unveiled directly by the central charges and internal symmetries. We next calculate from these symmetries first order fermionic and second order bosonic symmetries of the Lagrangian. We will notice that certain symmetries of the Lagrangian exist which are not revealed by this procedure. This is due to the absence of certain central charges in the algebra. We discuss the unique symmetries of $\mathcal{N} = 4$ SUSY-YM in Sections 3.1, 3.2 and 3.3 and the unique symmetries of $\mathcal{N} = 2$ FH in Section 3.4. In Appendix we list all symmetries unveiled by the procedure, including redundancies: symmetries which are the same from the Lagrangian perspective but which arise from different terms in the algebra as described in Section 2.4. No non-gauge symmetries of the $\mathcal{N} = 2$ vector multiplet are uncovered through this procedure as there are no central charges in this algebra.

3.1. $\mathcal{N} = 4$ SUSY-YM: First Order Bosonic Symmetries

The unique first order bosonic symmetries revealed by all the central charges and internal symmetries in this way are:

$$\delta_{BS1}^{(1)}(P) \begin{pmatrix} A^K \\ \mathrm{d} \end{pmatrix} \equiv P^K \begin{pmatrix} -\mathrm{d} \\ \Box A^K \end{pmatrix}, \quad \delta_{BS2}^{(1)}(Q) \begin{pmatrix} B^K \\ \mathrm{d} \end{pmatrix} \equiv Q^K \begin{pmatrix} -\mathrm{d} \\ \Box B^K \end{pmatrix} \qquad (42)$$

$$\delta_{BS3}^{(1)}(T) \begin{pmatrix} A^K \\ F^K \end{pmatrix} \equiv T^{KM} \begin{pmatrix} F^M \\ \Box A^M \end{pmatrix} \qquad (43)$$

$$\delta^{(1)}_{BS4}(T)\begin{pmatrix} B^K \\ G^K \end{pmatrix} \equiv T^{KM}\begin{pmatrix} G^M \\ \Box B^M \end{pmatrix} \tag{44}$$

$$\delta^{(1)}_{BS5}(W)\begin{pmatrix} A^J \\ G^J \end{pmatrix} \equiv W^{JK}\begin{pmatrix} G^K \\ \Box A^K \end{pmatrix} \tag{45}$$

$$\delta^{(1)}_{BS6}(V)\begin{pmatrix} F^J \\ G^J \end{pmatrix} \equiv (V^\mu)^{JK}\partial_\mu\begin{pmatrix} G^K \\ -F^K \end{pmatrix} \tag{46}$$

$$\delta^{(1)}_{BS7}(U)\begin{pmatrix} F^K \\ \mathrm{d} \end{pmatrix} \equiv (U^\mu)^{K}\partial_\mu\begin{pmatrix} \mathrm{d} \\ F^K \end{pmatrix} \tag{47}$$

$$\delta^{(1)}_{BS8}(U)\begin{pmatrix} G^K \\ A_\nu \end{pmatrix} \equiv (U^\mu)^{K}\begin{pmatrix} \partial^\nu F_{\mu\nu} \\ \eta_{\mu\nu}G^K \end{pmatrix} \tag{48}$$

$$\delta^{(1)}_{BS9}(Q)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv Q^K(\gamma^\mu)_c{}^d\partial_\mu\begin{pmatrix} \psi^K_d \\ -\lambda_d \end{pmatrix} \tag{49}$$

$$\delta^{(1)}_{BS10}(U)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv (U^\nu)^K(\gamma^5\gamma_\nu\gamma^\mu)_c{}^d\partial_\mu\begin{pmatrix} \psi^K_d \\ -\lambda_d \end{pmatrix} \tag{50}$$

$$\delta^{(1)}_{BS11}(P)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv P^K(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\begin{pmatrix} \psi^K_d \\ -\lambda_d \end{pmatrix} \tag{51}$$

$$\delta^{(1)}_{BS12}(W)\psi^K_c \equiv W^{KM}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\psi^M_d \tag{52}$$

$$\delta^{(1)}_{BS13}(V)\psi^K_c \equiv (V^\nu)^{KM}(\gamma^5\gamma_\nu\gamma^\mu)_c{}^d\partial_\mu\psi^M_d \tag{53}$$

$$\delta^{(1)}_{BS14}(T)\psi^K_c \equiv T^{KM}(\gamma^\mu)_c{}^d\partial_\mu\psi^M_d \tag{54}$$

along with the $U(1)$ gauge symmetries

$$\begin{aligned} \delta_G A_\nu &\equiv Q^K\partial_\nu A^K, \quad \delta_G A_\nu \equiv P^K\partial_\nu B^K, \\ \delta A_\nu &\equiv \varepsilon^a\chi^b_I C_{ab}\partial_\nu A^I, \quad \delta A_\nu \equiv \varepsilon^a\chi^b_I(\gamma^5)_{ab}\partial_\nu B^I \quad . \end{aligned} \tag{55}$$

The following identity proves useful in directly verifying these as Lagrangian symmetries:

$$(\gamma^5\gamma^{(\mu}\gamma_\alpha\gamma^{\nu)})^{(ab)} = 0 \tag{56}$$

where ( ) denotes symmetrization, *i.e.*, $(\gamma^\mu)^{(ab)} = (\gamma^\mu)^{ab} + (\gamma^\mu)^{ba}$.

It is interesting to note here that because of the absence of B^J to F^J coupling in the algebra, this method fails to uncover the first order bosonic symmetry of the Lagrangian

$$\delta^{(1)}_{BS15}(T)\begin{pmatrix} B^K \\ F^K \end{pmatrix} \equiv T^{KM}\begin{pmatrix} F^M \\ \Box B^M \end{pmatrix} \tag{57}$$

In addition, Lagrangian symmetries such as

$$\delta^{(1)}_{BS16}(U)\begin{pmatrix} G^K \\ \mathrm{d} \end{pmatrix} \equiv (U^\mu)^{K}\partial_\mu\begin{pmatrix} \mathrm{d} \\ G^K \end{pmatrix} \tag{58}$$

$$\delta^{(1)}_{BS17}(U)\begin{pmatrix} F^K \\ A_\nu \end{pmatrix} \equiv (U^\mu)^K \begin{pmatrix} \partial^\nu F_{\mu\nu} \\ \eta_{\mu\nu} F^K \end{pmatrix} \tag{59}$$

also are not manifest in the algebra. We will leave all such symmetries not manifested by the algebra out of the remaining calculations of second order bosonic and first order fermionic symmetries, as we are investigating how the absence of these symmetries fails to uncover further symmetries down the line.

3.2. Second Order Bosonic Symmetries

In Appendix A1, we list all the second order bosonic symmetries which are calculated in this way, including their redundancies. Here, we list only the unique symmetries, written in terms of the arbitrary matrices $(\Lambda_1)^{KJ}$, $(\Lambda_2^{\mu\nu})^{JK}$, $(\Lambda_3)^{IJ}$, $(\Lambda_4^\mu)^K$, Λ_5^K, and $(\Lambda_6^{\mu\nu})^J$:

$$\begin{gathered} \delta^{(2)}_{BS1}(\Lambda_1)A^K \equiv \Lambda_1^{[KJ]}\Box A^J, \quad \delta^{(2)}_{BS2}(\Lambda_1)B^K \equiv \Lambda_1^{[KJ]}\Box B^J \\ \delta^{(2)}_{BS3}(\Lambda_1)F^K \equiv \Lambda_1^{[KJ]}\Box F^J, \quad \delta^{(2)}_{BS4}(\Lambda_1)G^K \equiv \Lambda_1^{[KJ]}\Box G^K \\ \delta^{(2)}_{BS5}(\Lambda_2)F^J \equiv (\Lambda_2^{\mu\nu})^{[IJ]}\partial_\mu\partial_\nu F^I, \quad \delta^{(2)}_{BS6}(\Lambda_2)G^J \equiv (\Lambda_2^{\mu\nu})^{[IJ]}\partial_\mu\partial_\nu G^I \\ \delta^{(2)}_{BS7}(\Lambda_2)A_\nu \equiv \eta_{\nu\beta}(\Lambda_2^{[\mu\beta]})^{JJ}\partial^\alpha F_{\mu\alpha} \end{gathered} \tag{60}$$

$$\delta^{(2)}_{BS8}(\Lambda_1)\begin{pmatrix} A^K \\ B^K \end{pmatrix} \equiv \Lambda_1^{IJ}\begin{pmatrix} \delta^{IK}\Box B^J \\ -\delta^{JK}\Box A^I \end{pmatrix} \tag{61}$$

$$\delta^{(2)}_{BS9}(\Lambda_3)\begin{pmatrix} A^K \\ F^K \end{pmatrix} \equiv (\Lambda_3^\mu)^{IJ}\begin{pmatrix} \delta^{IK}\partial_\mu F^J \\ \delta^{JK}\partial_\mu\Box A^I \end{pmatrix} \tag{62}$$

$$\delta^{(2)}_{BS10}(\Lambda_3)\begin{pmatrix} B^K \\ F^K \end{pmatrix} \equiv (\Lambda_3^\mu)^{IJ}\begin{pmatrix} \delta^{IK}\partial_\mu F^J \\ \delta^{KJ}\partial_\mu\Box B^I \end{pmatrix} \tag{63}$$

$$\delta^{(2)}_{BS11}(\Lambda_3)\begin{pmatrix} A^J \\ G^J \end{pmatrix} \equiv (\Lambda_3^\mu)^{IK}\begin{pmatrix} \delta^{IJ}\partial_\mu G^K \\ \delta^{JK}\partial_\mu\Box A^I \end{pmatrix} \tag{64}$$

$$\delta^{(2)}_{BS12}(\Lambda_4)\begin{pmatrix} A^K \\ \mathrm{d} \end{pmatrix} \equiv (\Lambda_4^\mu)^K\begin{pmatrix} \partial_\mu \mathrm{d} \\ \partial_\mu\Box A^K \end{pmatrix} \tag{65}$$

$$\delta^{(2)}_{BS13}(\Lambda_4)\begin{pmatrix} A^J \\ A_\nu \end{pmatrix} \equiv (\Lambda_4^\mu)^J\begin{pmatrix} \partial^\nu F_{\mu\nu} \\ \eta_{\mu\nu}\Box A^J \end{pmatrix} \tag{66}$$

$$\delta^{(2)}_{BS14}(\Lambda_4)\begin{pmatrix} B^J \\ A_\nu \end{pmatrix} \equiv (\Lambda_4^\mu)^J\begin{pmatrix} \partial^\nu F_{\mu\nu} \\ \eta_{\mu\nu}\Box B^J \end{pmatrix} \tag{67}$$

$$\delta^{(2)}_{BS15}(\Lambda_5)\begin{pmatrix} F^K \\ \mathrm{d} \end{pmatrix} \equiv \Lambda_5^K\begin{pmatrix} \Box \mathrm{d} \\ -\Box F^K \end{pmatrix} \tag{68}$$

$$\delta^{(2)}_{BS16}(\Lambda_5)\begin{pmatrix} G^K \\ \mathrm{d} \end{pmatrix} \equiv \Lambda_5^K\begin{pmatrix} \Box \mathrm{d} \\ -\Box G^K \end{pmatrix} \tag{69}$$

$$\delta^{(2)}_{BS17}(\Lambda_6)\begin{pmatrix} G^J \\ \mathrm{d} \end{pmatrix} \equiv (\Lambda_6^{\mu\nu})^J \begin{pmatrix} \partial_\mu\partial_\nu \mathrm{d} \\ -\partial_\mu\partial_\nu G^J \end{pmatrix} \tag{70}$$

$$\delta^{(2)}_{BS18}(\Lambda_1)\begin{pmatrix} F^J \\ G^J \end{pmatrix} \equiv \Lambda_1^{IK} \begin{pmatrix} \delta^{IJ}\Box G^K \\ -\Box\delta^{JK} F^I \end{pmatrix} \tag{71}$$

$$\delta^{(2)}_{BS19}(\Lambda_6)\begin{pmatrix} F^J \\ A_\alpha \end{pmatrix} \equiv (\Lambda_6^{\mu\nu})^J (U,V) \begin{pmatrix} \partial_\nu\partial^\alpha F_{\mu\alpha} \\ -\eta_{\mu\alpha}\partial_\nu F^J \end{pmatrix} \tag{72}$$

and

$$\delta^{(2)}_{BS20}(\Lambda_1)\psi^K_c \equiv \Lambda_1^{[JK]}\Box\psi^J_c \tag{73}$$

$$\delta^{(2)}_{BS21}(\Lambda_2)\psi^K_c \equiv [(\Lambda_2^{\rho\sigma})^{KJ} - (\Lambda_2^{\sigma\rho})^{JK}](\gamma_\rho\gamma^\mu\gamma_\sigma\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^J_d \tag{74}$$

$$\delta^{(2)}_{BS22}(\Lambda_2)\lambda_c \equiv (\Lambda_2^{[\mu\nu]})^{KK}(\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\lambda_d, \tag{75}$$

$$\delta^{(2)}_{BS23}(\Lambda_3)\psi^K_c \equiv (\Lambda_3^\mu)^{[JK]}(\gamma^5\gamma_\mu)_c{}^d\Box\psi^J_d \tag{76}$$

$$\delta^{(2)}_{BS24}(\Lambda_3)\lambda_c \equiv (\Lambda_3^\nu)^{KK}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\partial_\nu\lambda_d \tag{77}$$

$$\delta^{(2)}_{BS25}(\Lambda_3)\psi^K_c \equiv (\Lambda_3^\mu)^{KJ}(\gamma^5\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^J_d \tag{78}$$

$$\delta^{(2)}_{BS26}(\Lambda_3)\psi^K_c \equiv (\Lambda_3^\mu)^{[JK]}(\gamma_\mu)_c{}^d\Box\psi^J_d + 2(\Lambda_3^\mu)^{KJ}(\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^J_d \tag{79}$$

$$\delta^{(2)}_{BS27}(\Lambda_3)\lambda_c \equiv (\Lambda_3^\mu)^{KK}(\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\lambda_d \tag{80}$$

$$\delta^{(2)}_{BS28}(\Lambda_1)\psi^K_c \equiv \Lambda_1^{KJ}(\gamma^5)_c{}^d\Box\psi^J_d \tag{81}$$

$$\delta^{(2)}_{BS29}(\Lambda_1)\lambda_c \equiv \Lambda_1^{KK}(\gamma^5)_c{}^d\Box\lambda_d \tag{82}$$

$$\delta^{(2)}_{BS30}(\Lambda_5)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv \Lambda_5^K(\gamma^5)_c{}^d \begin{pmatrix} \Box\psi^K_d \\ \Box\lambda_d \end{pmatrix} \tag{83}$$

$$\delta^{(2)}_{BS31}(\Lambda_5)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv \Lambda_5^K \begin{pmatrix} \Box\psi^K_c \\ -\Box\lambda_c \end{pmatrix} \tag{84}$$

$$\delta^{(2)}_{BS32}(\Lambda_4)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv (\Lambda_4^\alpha)^K \begin{pmatrix} (\gamma^5\gamma^\nu\gamma_\alpha\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^K_d \\ (\gamma^5\gamma_\alpha)_c{}^d\Box\lambda_d \end{pmatrix} \tag{85}$$

$$\delta^{(2)}_{BS33}(\Lambda_4)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv (\Lambda_4^\mu)^K \begin{pmatrix} (\gamma^5\gamma_\mu)_c{}^d\Box\psi^K_d \\ (\gamma^5\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\lambda_d \end{pmatrix} \tag{86}$$

$$\delta^{(2)}_{BS34}(\Lambda_4)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv (\Lambda_4^\mu)^K \begin{pmatrix} (\gamma_\mu)_c{}^d\Box\psi^K_d \\ (\gamma^\nu\gamma_\mu\gamma^\alpha)_c{}^d\partial_\nu\partial_\alpha\lambda_d \end{pmatrix} \tag{87}$$

$$\delta^{(2)}_{BS35}(\Lambda_4)\begin{pmatrix} \lambda_c \\ \psi^K_c \end{pmatrix} \equiv (\Lambda_4^\mu)^K \begin{pmatrix} (\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\psi^K_d \\ (\gamma_\mu)_c{}^d\Box\lambda_d \end{pmatrix}, \tag{88}$$

$$\delta^{(2)}_{BS36}(\Lambda_6)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix}\equiv(\Lambda_6^{\mu\nu})^K\begin{pmatrix}(\gamma_\nu\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\psi^K_d\\ -(\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\lambda_d\end{pmatrix}, \tag{89}$$

This analysis seems to not miss any second order bosonic symmetries which act on the fermions λ_a and ψ^J_a. However, the missing first order bosonic symmetries alluded to previously which act on the bosons clearly manifest themselves here in missing second order bosonic symmetries. Basically, as the fields A^J and B^J enter the Lagrangian in the same way, they should have the same first and second order symmetries. The same should hold for F^J and G^J. But clearly since, for example, the algebra is *not* symmetric between exchange of $A^J\leftrightarrow B^J$ or $F^J\leftrightarrow G^J$, Lagrangian symmetries involving these field pairs will be missed when generated from the algebra in the manner presented here.

3.3. $\mathcal{N}=4$ SUSY-YM: First Order Fermionic Symmetries

All such possible calculations are listed in the Appendix A2, some of which are redundant as in the second order bosonic case. Here is listed only the unique symmetries.

$$\delta^{(1)}_{FS1}(P)\begin{pmatrix}A^K\\ \psi^J_b\end{pmatrix}\equiv\varepsilon^a_JP^K\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^J_b\\ (\gamma^5)_{ab}\Box A^K\end{pmatrix} \tag{90}$$

$$\delta^{(1)}_{FS2}(P)\begin{pmatrix}A^J\\ \lambda_b\end{pmatrix}\equiv\varepsilon^aP^J\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b\\ (\gamma^5)_{ab}\Box A^J\end{pmatrix} \tag{91}$$

$$\delta^{(1)}_{FS3}(Q)\begin{pmatrix}B^J\\ \lambda_b\end{pmatrix}\equiv\varepsilon^aQ^J\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b\\ (\gamma^5)_{ab}\Box B^J\end{pmatrix} \tag{92}$$

$$\delta^{(1)}_{FS4}(Q)\begin{pmatrix}B^K\\ \psi^I_b\end{pmatrix}\equiv\varepsilon^a_IQ^K\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^I_b\\ (\gamma^5)_{ab}\Box B^K\end{pmatrix} \tag{93}$$

$$\delta^{(1)}_{FS5}(P)\begin{pmatrix}B^K\\ \lambda_b\end{pmatrix}\equiv\varepsilon^aP^K\begin{pmatrix}i(\gamma^\mu)_a{}^b\partial_\mu\lambda_b\\ C_{ab}\Box B^K\end{pmatrix} \tag{94}$$

$$\delta^{(1)}_{FS6}(P)\begin{pmatrix}B^J\\ \psi^K_b\end{pmatrix}\equiv\varepsilon^a_JP^K\begin{pmatrix}i(\gamma^\mu)_a{}^b\partial_\mu\psi^K_b\\ C_{ab}\Box B^J\end{pmatrix} \tag{95}$$

$$\delta^{(1)}_{FS7}(Q)\begin{pmatrix}A^J\\ \lambda_b\end{pmatrix}\equiv\varepsilon^aQ^J\begin{pmatrix}(\gamma^\mu)_a{}^b\partial_\mu\lambda_b\\ -iC_{ab}\Box A^J\end{pmatrix} \tag{96}$$

$$\delta^{(1)}_{FS8}(Q)\begin{pmatrix}A^I\\ \psi^K_b\end{pmatrix}\equiv\varepsilon^a_IQ^K\begin{pmatrix}(\gamma^\mu)_a{}^b\partial_\mu\psi^K_b\\ -iC_{ab}\Box A^I\end{pmatrix} \tag{97}$$

and

$$\delta^{(1)}_{FS9}(P)\begin{pmatrix}F^K\\ \lambda_b\end{pmatrix}\equiv\varepsilon^aP^K\begin{pmatrix}(\gamma^5)_a{}^b\Box\lambda_b\\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^K\end{pmatrix} \tag{98}$$

$$\delta^{(1)}_{FS10}(P)\begin{pmatrix}F^J\\ \psi^K_b\end{pmatrix}\equiv\varepsilon^a_JP^K\begin{pmatrix}(\gamma^5)_a{}^b\Box\psi^K_b\\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^J\end{pmatrix} \tag{99}$$

$$\delta^{(1)}_{FS11}(Q)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a Q^J \begin{pmatrix} (\gamma^5)_a{}^b \Box \lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \tag{100}$$

$$\delta^{(1)}_{FS12}(Q)\begin{pmatrix} G^I \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_I Q^K \begin{pmatrix} (\gamma^5)_a{}^b \Box \psi^K_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^I \end{pmatrix} \tag{101}$$

$$\delta^{(1)}_{FS13}(Q)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a Q^J \begin{pmatrix} i(\gamma^5)_a{}^b \Box \psi^J_b \\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{102}$$

$$\delta^{(1)}_{FS14}(Q)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I Q^I \begin{pmatrix} i(\gamma^5)_a{}^b \Box \lambda_b \\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{103}$$

and

$$\delta^{(1)}_{FS15}(Q)\begin{pmatrix} F^I \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_I Q^K \begin{pmatrix} \Box \psi^K_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^I \end{pmatrix} \tag{104}$$

$$\delta^{(1)}_{FS16}(Q)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a Q^J \begin{pmatrix} \Box \lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \tag{105}$$

$$\delta^{(1)}_{FS17}(P)\begin{pmatrix} G^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} \Box \lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^K \end{pmatrix} \tag{106}$$

$$\delta^{(1)}_{FS18}(P)\begin{pmatrix} G^J \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_J P^K \begin{pmatrix} \Box \psi^K_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \tag{107}$$

$$\delta^{(1)}_{FS19}(P)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a P^J \begin{pmatrix} \Box \psi^J_a \\ i(\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{108}$$

$$\delta^{(1)}_{FS20}(P)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I P^I \begin{pmatrix} \Box \lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{109}$$

and

$$\delta^{(1)}_{FS21}(T)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM} \begin{pmatrix} (\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ iC_{ab}\Box A^M \end{pmatrix} \tag{110}$$

$$\delta^{(1)}_{FS22}(W)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM} \begin{pmatrix} (\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ iC_{ab}\Box B^M \end{pmatrix} \tag{111}$$

$$\delta^{(1)}_{FS23}(T)\begin{pmatrix} A^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I T^{IK} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ C_{ab}\Box A^K \end{pmatrix} \tag{112}$$

$$\delta^{(1)}_{FS24}(T)\begin{pmatrix} A^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \psi^J_b \\ C_{ab}\Box A^M \end{pmatrix} \tag{113}$$

$$\delta^{(1)}_{FS25}(T)\begin{pmatrix} A^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ C_{ab}\Box A^J \end{pmatrix} \tag{114}$$

$$\delta^{(1)}_{FS26}(W)\begin{pmatrix} B^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} W^{KM} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ C_{ab}\Box B^J \end{pmatrix} \tag{115}$$

and

$$\delta^{(1)}_{FS27}(W)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} (\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box A^M \end{pmatrix} \tag{116}$$

$$\delta^{(1)}_{FS28}(T)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} (\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box B^M \end{pmatrix} \tag{117}$$

$$\delta^{(1)}_{FS29}(W)\begin{pmatrix} A^M \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I W^{IM}\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box A^M \end{pmatrix} \tag{118}$$

$$\delta^{(1)}_{FS30}(W)\begin{pmatrix} A^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} W^{KM}\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ (\gamma^5)_{ab}\Box A^J \end{pmatrix} \tag{119}$$

$$\delta^{(1)}_{FS31}(W)\begin{pmatrix} A^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} W^{KM}\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^J_b \\ (\gamma^5)_{ab}\Box A^M \end{pmatrix} \tag{120}$$

$$\delta^{(1)}_{FS32}(T)\begin{pmatrix} B^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I T^{IK}\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box B^K \end{pmatrix} \tag{121}$$

$$\delta^{(1)}_{FS33}(T)\begin{pmatrix} B^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM}\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ (\gamma^5)_{ab}\Box B^J \end{pmatrix} \tag{122}$$

$$\delta^{(1)}_{FS34}(T)\begin{pmatrix} B^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM}\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^J_b \\ (\gamma^5)_{ab}\Box B^M \end{pmatrix} \tag{123}$$

and

$$\delta^{(1)}_{FS35}(T)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^M_b \\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{124}$$

$$\delta^{(1)}_{FS36}(W)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^M_b \\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu F^M \end{pmatrix} \tag{125}$$

$$\delta^{(1)}_{FS37}(W)\begin{pmatrix} F^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} W^{KM}\begin{pmatrix} (\gamma^5)_a{}^b\Box\psi^M_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \tag{126}$$

$$\delta^{(1)}_{FS38}(T)\begin{pmatrix} G^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I T^{IK}\begin{pmatrix} (\gamma^5)_a{}^b\Box\lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^K \end{pmatrix} \tag{127}$$

$$\delta^{(1)}_{FS39}(T)\begin{pmatrix} G^K \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM}\begin{pmatrix} (\gamma^5)_a{}^b\Box\psi^N_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{128}$$

$$\delta^{(1)}_{FS40}(T)\begin{pmatrix} G^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM}\begin{pmatrix} (\gamma^5)_a{}^b\Box\psi^M_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \tag{129}$$

$$\delta^{(1)}_{FS41}(T)\begin{pmatrix} \mathrm{d} \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I T^{IM}\begin{pmatrix} (\gamma^5)_a{}^b\Box\psi^M_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{130}$$

and

$$\delta^{(1)}_{FS42}(V)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\mu)^{JM} \begin{pmatrix} i(\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ (\gamma^5)_{ab}\partial_\mu F^M \end{pmatrix} \tag{131}$$

$$\delta^{(1)}_{FS43}(V)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM} \begin{pmatrix} (\gamma^5\gamma^\nu\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^M \end{pmatrix} \tag{132}$$

$$\delta^{(1)}_{FS44}(V)\begin{pmatrix} F^M \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (V^\mu)^{IM} \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ i(\gamma^5)_{ab}\partial_\mu F^M \end{pmatrix} \tag{133}$$

$$\delta^{(1)}_{FS45}(V)\begin{pmatrix} F^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} (V^\mu)^{KM} \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^J_b \\ i(\gamma^5)_{ab}\partial_\mu F^M \end{pmatrix} \tag{134}$$

$$\delta^{(1)}_{FS46}(U)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\mu)^{J} \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ i(\gamma^5)_{ab}\partial_\mu F^J \end{pmatrix} \tag{135}$$

$$\delta^{(1)}_{FS47}(U)\begin{pmatrix} F^K \\ \psi^I_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^{K} \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^I_b \\ i(\gamma^5)_{ab}\partial_\mu F^K \end{pmatrix} \tag{136}$$

$$\delta^{(1)}_{FS48}(V)\begin{pmatrix} F^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} (\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ -i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^J \end{pmatrix} \tag{137}$$

$$\delta^{(1)}_{FS49}(U)\begin{pmatrix} F^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^{K} \begin{pmatrix} (\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ -i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^K \end{pmatrix} \tag{138}$$

$$\delta^{(1)}_{FS50}(U)\begin{pmatrix} F^J \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_J (U^\rho)^{K} \begin{pmatrix} (\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^K_b \\ -i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^J \end{pmatrix} \tag{139}$$

and

$$\delta^{(1)}_{FS51}(U)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\mu)^{J} \begin{pmatrix} \partial^\nu\partial_{[\mu}(\gamma_{\nu]})_a{}^b\lambda_b \\ (\sigma^\nu{}_\mu)_{ab}\partial_\nu G^J \end{pmatrix} \tag{140}$$

$$\delta^{(1)}_{FS52}(U)\begin{pmatrix} G^K \\ \psi^I_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^{K} \begin{pmatrix} \partial^\nu\partial_{[\mu}(\gamma_{\nu]})_a{}^b\psi^I_b \\ (\sigma^\nu{}_\mu)_{ab}\partial_\nu G^K \end{pmatrix} \tag{141}$$

$$\delta^{(1)}_{FS53}(U)\begin{pmatrix} G^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^{K} \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^K \end{pmatrix} \tag{142}$$

$$\delta^{(1)}_{FS54}(U)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (U^\mu)^{J} \begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^J_b \\ iC_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{143}$$

$$\delta^{(1)}_{FS55}(U)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^{I} \begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ iC_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{144}$$

$$\delta^{(1)}_{FS56}(V)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM} \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ -(\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^M \end{pmatrix} \tag{145}$$

$$\delta^{(1)}_{FS57}(V)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\mu)^{JM} \begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ -iC_{ab}\partial_\mu G^M \end{pmatrix} \tag{146}$$

$$\delta^{(1)}_{FS58}(U)\begin{pmatrix} G^J \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_J (U^\rho)^K \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^K_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\nu G^J \end{pmatrix} \tag{147}$$

$$\delta^{(1)}_{FS59}(V)\begin{pmatrix} G^M \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (V^\mu)^{IM} \begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ iC_{ab}\partial_\mu G^M \end{pmatrix} \tag{148}$$

$$\delta^{(1)}_{FS60}(V)\begin{pmatrix} G^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} (V^\mu)^{KM} \begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^N_a \\ iC_{ab}\partial_\mu G^M \end{pmatrix} \tag{149}$$

$$\delta^{(1)}_{FS61}(V)\begin{pmatrix} G^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^J \end{pmatrix} \tag{150}$$

$$\delta^{(1)}_{FS62}(U)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^I \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu \mathrm{d} \end{pmatrix} \tag{151}$$

$$\delta^{(1)}_{FS63}(V)\begin{pmatrix} \mathrm{d} \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I (V^\rho)^{IM} \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu \mathrm{d} \end{pmatrix} \tag{152}$$

and

$$\delta^{(1)}_{FS64}(P)\begin{pmatrix} A_\mu \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I P^I \begin{pmatrix} (\gamma^5\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\lambda_b \\ -i(\gamma^5\gamma^\beta)_{ab}\partial^\alpha F_{\alpha\beta} \end{pmatrix} \tag{153}$$

$$\delta^{(1)}_{FS65}(P)\begin{pmatrix} A_\mu \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} (\gamma^5\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\psi^K_b \\ -i(\gamma^5\gamma^\beta)_{ab}\partial^\alpha F_{\alpha\beta} \end{pmatrix} \tag{154}$$

$$\delta^{(1)}_{FS66}(Q)\begin{pmatrix} A_\mu \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a Q^J \begin{pmatrix} -(\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\psi^J_b \\ \frac{1}{2}(\gamma^\alpha\sigma^{\mu\nu})_{ba}\partial_\alpha F_{\mu\nu} \end{pmatrix} \tag{155}$$

$$\delta^{(1)}_{FS67}(Q)\begin{pmatrix} A_\mu \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I Q^I \begin{pmatrix} -(\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\lambda_b \\ \frac{1}{2}(\gamma^\alpha\sigma^{\mu\nu})_{ba}\partial_\alpha F_{\mu\nu} \end{pmatrix} \tag{156}$$

$$\delta^{(1)}_{FS68}(T)\begin{pmatrix} A_\mu \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I T^{IM} \begin{pmatrix} (\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\psi^M_b \\ -\frac{1}{2}(\gamma^\alpha\sigma^{\mu\nu})_{ba}\partial_\alpha F_{\mu\nu} \end{pmatrix} \tag{157}$$

$$\delta^{(1)}_{FS69}(W)\begin{pmatrix} A_\mu \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I W^{IM} \begin{pmatrix} -(\gamma^5\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\psi^M_b \\ -\frac{1}{2}(\gamma^5\gamma^\alpha\sigma^{\mu\nu})_{ba}\partial_\alpha F_{\mu\nu} \end{pmatrix} \tag{158}$$

$$\delta^{(1)}_{FS70}(U)\begin{pmatrix} A_\mu \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (U_\mu)^J \begin{pmatrix} i(\gamma^5\gamma^\nu)_a{}^b\partial_\nu\psi^J_b \\ -(\gamma^5)_{ab}\partial_\nu F^{\mu\nu} \end{pmatrix} \tag{159}$$

$$\delta^{(1)}_{FS71}(U)\begin{pmatrix} A_\mu \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U_\mu)^I \begin{pmatrix} -i(\gamma^5\gamma^\nu)_a{}^b\partial_\nu\lambda_b \\ (\gamma^5)_{ab}\partial_\nu F^{\mu\nu} \end{pmatrix} \tag{160}$$

$$\delta^{(1)}_{FS72}(U)\begin{pmatrix} A_\mu \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^I \begin{pmatrix} (\gamma^5\gamma_\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\nu\lambda_b \\ \frac{1}{2}(\gamma^5\gamma_\rho\gamma^\nu\sigma^{\alpha\beta})_{ba}\partial_\nu F_{\alpha\beta} \end{pmatrix} \tag{161}$$

$$\delta^{(1)}_{FS73}(V)\begin{pmatrix} A_\mu \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I (V^\rho)^{IM} \begin{pmatrix} (\gamma^5\gamma_\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\nu\psi^M_b \\ \frac{1}{2}(\gamma^5\gamma_\rho\gamma^\nu\sigma^{\alpha\beta})_{ba}\partial_\nu F_{\alpha\beta} \end{pmatrix} \tag{162}$$

and

$$\delta^{(1)}_{FS74}(U)\begin{pmatrix} A^J \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_J (U^\rho)^K \begin{pmatrix} -(\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^K_b \\ i(\gamma^5\gamma_\rho)_{ab}\Box A^J \end{pmatrix} \tag{163}$$

$$\delta^{(1)}_{FS75}(V)\begin{pmatrix} A^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} -(\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5\gamma_\rho)_{ab}\Box A^J \end{pmatrix} \tag{164}$$

$$\delta^{(1)}_{FS76}(V)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM} \begin{pmatrix} (\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5\gamma_\rho)_{ab}\Box A^M \end{pmatrix} \tag{165}$$

$$\delta^{(1)}_{FS77}(U)\begin{pmatrix} A^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} (\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ -i(\gamma^5\gamma_\rho)_{ab}\Box A^K \end{pmatrix} \tag{166}$$

$$\delta^{(1)}_{FS78}(U)\begin{pmatrix} B^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma_\rho)_{ab}\Box B^K \end{pmatrix} \tag{167}$$

$$\delta^{(1)}_{FS79}(V)\begin{pmatrix} B^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ (\gamma_\rho)_{ab}\Box B^J \end{pmatrix} \tag{168}$$

$$\delta^{(1)}_{FS80}(U)\begin{pmatrix} B^J \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_J (U^\rho)^K \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^K_b \\ (\gamma_\rho)_{ab}\Box B^J \end{pmatrix} \tag{169}$$

$$\delta^{(1)}_{FS81}(V)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM} \begin{pmatrix} -i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ (\gamma_\rho)_{ab}\Box B^M \end{pmatrix} \tag{170}$$

and

$$\delta^{(1)}_{FS82}(T)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM} \begin{pmatrix} i\Box\psi^M_a \\ (\gamma^\mu)_{ab}\partial_\mu F^M \end{pmatrix} \tag{171}$$

$$\delta^{(1)}_{FS83}(W)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM} \begin{pmatrix} i\Box\psi^M_a \\ (\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{172}$$

$$\delta^{(1)}_{FS84}(T)\begin{pmatrix} F^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I T^{IK} \begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^K \end{pmatrix} \tag{173}$$

$$\delta^{(1)}_{FS85}(T)\begin{pmatrix} F^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} \Box\psi^M_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \tag{174}$$

$$\delta^{(1)}_{FS86}(T)\begin{pmatrix} F^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} \Box\psi^J_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^M \end{pmatrix} \tag{175}$$

$$\delta^{(1)}_{FS87}(W)\begin{pmatrix} G^M \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I W^{IM} \begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{176}$$

$$\delta^{(1)}_{FS88}(W)\begin{pmatrix} G^J \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} W^{KM} \begin{pmatrix} \Box\psi^M_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \tag{177}$$

$$\delta^{(1)}_{FS89}(W)\begin{pmatrix} G^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I \epsilon^{IJK} W^{KM} \begin{pmatrix} \Box\psi^J_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{178}$$

$$\delta^{(1)}_{FS90}(W)\begin{pmatrix} \mathrm{d} \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I W^{IM} \begin{pmatrix} \Box\psi^M_a \\ i(\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \tag{179}$$

3.4. Symmetries of the $\mathcal{N}=2$ FH Lagrangian

The symmetries of the $\mathcal{N}=2$ FH system follow analogously from the $\mathcal{N}=4$ calculations. The first order bosonic symmetries of the $\mathcal{N}=2$ FH system calculated from the central charges and internal symmetries are

$$\tilde{\delta}^{(1)}_{BS1}(\tilde{T})\begin{pmatrix} \tilde{A}^k \\ \tilde{F}^k \end{pmatrix} \equiv \tilde{T}^{km} \begin{pmatrix} \tilde{F}^m \\ \Box\tilde{A}^m \end{pmatrix} \tag{180}$$

$$\tilde{\delta}^{(1)}_{BS2}(\tilde{T})\begin{pmatrix} \tilde{B}^k \\ \tilde{G}^k \end{pmatrix} \equiv \tilde{T}^{km} \begin{pmatrix} \tilde{G}^m \\ \Box\tilde{B}^m \end{pmatrix} \tag{181}$$

$$\tilde{\delta}^{(1)}_{BS3}(\tilde{T})\tilde{\psi}^k_c \equiv \tilde{T}^{km}(\gamma^\mu)_c{}^d\partial_\mu\tilde{\psi}^m_d \tag{182}$$

with

$$\tilde{T}^{km} \equiv \tilde{R}^{ijkm} C_{ab}\varepsilon^a_i\chi^b_j \tag{183}$$

where $i,j,k,m=1,2$, and ε^a_i and χ^b_j are once again infinitesimal Grassmann spinors. Here, we clearly notice the absence of symmetries between $A^J \leftrightarrow B^J$, $A^J \leftrightarrow G^J$, $B^J \leftrightarrow F^J$, and $G^J \leftrightarrow F^J$. As in the $\mathcal{N}=4$ case, this is a direct result of the absence of coupling terms between these fields in the algebra.

Interestingly, we find that the second order bosonic symmetries calculated from these first order symmetries all vanish identically

$$\tilde{\delta}^{(2)}_{BS1}(\tilde{T}_1,\tilde{T}_2)\tilde{A}^k \equiv [\tilde{\delta}^{(1)}_{BS1}(\tilde{T}_1),\tilde{\delta}^{(1)}_{BS1}(\tilde{T}_2)]\tilde{A}^k = \tilde{\Lambda}^{jk}_1(\tilde{T}_1,\tilde{T}_2)\Box\tilde{A}^j = 0 \tag{184}$$

$$\tilde{\delta}^{(2)}_{BS2}(\tilde{T}_1,\tilde{T}_2)\tilde{B}^k \equiv [\tilde{\delta}^{(1)}_{BS2}(\tilde{T}_1),\tilde{\delta}^{(1)}_{BS2}(\tilde{T}_2)]\tilde{B}^k = \tilde{\Lambda}^{jk}_1(\tilde{T}_1,\tilde{T}_2)\Box\tilde{B}^j = 0 \tag{185}$$

$$\tilde{\delta}^{(2)}_{BS3}(\tilde{T}_1,\tilde{T}_2)\tilde{F}^k \equiv [\tilde{\delta}^{(1)}_{BS1}(\tilde{T}_1),\tilde{\delta}^{(1)}_{BS1}(\tilde{T}_2)]\tilde{F}^k = \tilde{\Lambda}^{jk}_1(\tilde{T}_1,\tilde{T}_2)\Box\tilde{F}^j = 0 \tag{186}$$

$$\tilde{\delta}^{(2)}_{BS4}(\tilde{T}_1,\tilde{T}_2)\tilde{G}^k \equiv [\tilde{\delta}^{(1)}_{BS2}(T_1),\tilde{\delta}^{(1)}_{BS2}(\tilde{T}_2)]\tilde{G}^k = \tilde{\Lambda}^{jk}_1(\tilde{T}_1,\tilde{T}_2)\Box\tilde{G}^j = 0 \tag{187}$$

$$\tilde{\delta}^{(2)}_{BS5}(\tilde{T}_1,\tilde{T}_2)\tilde{\psi}^k_c \equiv [\tilde{\delta}^{(1)}_{BS3}(\tilde{T}_1),\tilde{\delta}^{(1)}_{BS3}(\tilde{T}_2)]\tilde{\psi}^k_c = \tilde{\Lambda}^{jk}_1(\tilde{T}_1,\tilde{T}_2)\Box\tilde{\psi}^J_c = 0 \tag{188}$$

as

$$\tilde{\Lambda}^{jk}_{1,1}(\tilde{T}_1,\tilde{T}_2) \equiv \tilde{T}^{jm}_{[1}\tilde{T}^{mk}_{2]} = 0, \quad j,k,m=1,2 \tag{189}$$

even though for a general matrix $\tilde{\Lambda}^{jk}$,

$$\begin{aligned} \tilde{\delta}^{(2)}_{BS}\tilde{\chi}^j_C &\equiv \tilde{\Lambda}^{[jk]}\Box\tilde{\chi}^k_C, \\ \tilde{\chi}^j_C &\equiv (\tilde{A}^j,\tilde{B}^j,\tilde{F}^j,\tilde{G}^j,\tilde{\psi}^j_c) \end{aligned} \tag{190}$$

is still a symmetry of the $\mathcal{N}=2$ FH Lagrangian.

On the other hand, several first order fermionic symmetries still remain after reduction to the $\mathcal{N} = 2$ FH system:

$$\begin{aligned}
\tilde{\delta}^{(1)}_{FS1}(\tilde{T})\begin{pmatrix}\tilde{A}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i\tilde{T}^{ik}\begin{pmatrix}-(\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^1_b\\ iC_{ab}\Box\tilde{A}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS2}(\tilde{T})\begin{pmatrix}\tilde{F}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i\tilde{T}^{ik}\begin{pmatrix}\Box\tilde{\psi}^1_a\\ i(\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS3}(\tilde{T})\begin{pmatrix}\tilde{A}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^2)^{ij}\tilde{T}^{jk}\begin{pmatrix}i(\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^2_b\\ C_{ab}\Box\tilde{A}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS4}(\tilde{T})\begin{pmatrix}\tilde{F}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^2)^{ij}T^{jk}\begin{pmatrix}-i\Box\tilde{\psi}^2_a\\ (\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS5}(\tilde{T})\begin{pmatrix}\tilde{B}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^3)^{ij}\tilde{T}^{jk}\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^1_b\\ (\gamma^5)_{ab}\Box\tilde{B}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS6}(\tilde{T})\begin{pmatrix}\tilde{G}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^3)^{ij}\tilde{T}^{jk}\begin{pmatrix}-i(\gamma^5)_a{}^b\Box\tilde{\psi}^1_b\\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS7}(\tilde{T})\begin{pmatrix}\tilde{B}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^1)^{ij}\tilde{T}^{jk}\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^2_b\\ (\gamma^5)_{ab}\Box\tilde{B}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS8}(\tilde{T})\begin{pmatrix}\tilde{G}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^1)^{ij}\tilde{T}^{jk}\begin{pmatrix}-i(\gamma^5)_a{}^b\Box\tilde{\psi}^2_b\\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k\end{pmatrix}
\end{aligned} \tag{191}$$

These are only the unique symmetries uncovered via this method, the redundant calculations being shown once again in Appendix A3. Here we notice as in the bosonic case, that these fermionic symmetries are not themselves symmetric with respect to $A^J \leftrightarrow B^J$ and $F^J \leftrightarrow G^J$. Again, this is a direct result of the absence of the corresponding central charge or internal symmetry in the algebra.

4. Discussion

The $d = 4$, $\mathcal{N} = 4$ SUSY-YM system is important to many theoretical models in physics today. As it is a conformal field theory, it's possible that its study can lead to further understanding of "walking" theories such as technicolor. In string theory, the AdS/CFT correspondence relates calculations of $d = 4$, $\mathcal{N} = 4$ SUSY-YM to classical supergravity calculations on $AdS_5 \times S^5$, where the correspondence is weak to strong and vice versa. In an effort to more accurately describe the standard model, this has been taken further to include correspondences to gauge theories with running couplings. Even so, the problem of how to augment the dynamical theory of $d = 4$, $\mathcal{N} = 4$ SUSY-YM with a *finite* number of auxiliary fields such that the algebra closes has been unsolved for quite some time. A solution to this problem would be helpful to more fully understand these aforementioned theories relating to conformal field theories.

In this paper, we chose a particular set of auxiliary fields for $d = 4$, $\mathcal{N} = 4$ SUSY-YM and catalogued the Lagrangian symmetries manifest in the central charges and internal symmetries of the resulting algebra. It was noted how not all possible Lagrangian symmetries can be uncovered this way, as certain central charges and internal symmetries are missing from the algebra. We reinforce here that all results presented are from straightforward, actual calculations with no assumptions of centrality. For instance, we have directly calculated that the SUSY-YM Lagrangian in Equation (4) is invariant with respect to the transformation laws in Equations (5)–(8). We have directly calculated that these transformation laws satisfy the anti-commutation relations in Equations (11)–(15). The main result of this paper is how these transformation laws and anti-commutators lead by direct calculation to the first and second order Lagrangian symmetries presented in Section 3.

Furthermore, reduction of this particular $\mathcal{N} = 4$ system to the $\mathcal{N} = 2$ Fayet hypermultiplet and $\mathcal{N} = 2$ vector multiplet was shown to follow from our direct calculations. Here it was noticed how in this reduction, central charges and internal symmetries are lost from the algebra. In the case of the vector multiplet, all charges and internal symmetries are lost as the algebra closes. In the case of the Fayet hypermultiplet, some central charges and internal symmetries remain, as this algebra does not close.

Finally, we make a note on quantization of non-closed systems such as the $\mathcal{N} = 4$ SUSY-YM system investigated in detail in this paper. In general, non-closure of an algebra leads to an added difficulty in the quantization procedure. Perhaps the most ubiquitous example is the criticality of string theory. For quantum non-critical strings, one must solve the Liouville theory. This is not necessary in the case of critical strings [26,27]. In the case of our results of the $\mathcal{N} = 4$ SUSY-YM system, we have laid out our results in the hopes of eventually obtaining a closed system, in the sense of Equation (1), without an infinite number of auxiliary fields. For instead quantization of the non-closed system presented, the specific forms of the non-closure terms we calculated are important in the same vein as the Liouville theory for non-critical strings. We leave this quantization as a future project.

> *"It is while you are patiently toiling at the little tasks of life that the meaning and shape of the great whole of life dawn on you."*
>
> —Phillips Brooks

Acknowledgments

This research was supported in part by the endowment of the John S. Toll Professorship, the University of Maryland Center for String & Particle Theory, National Science Foundation Grant PHY-0354401. Sylvester James Gates, Jr. & Kory Stiffler offer additional gratitude to the Martin L. King. Visiting Professorship program and to the Massachusetts Institute of Technology Center for Theoretical Physics for support and hospitality extended during the undertaking of this work. We would also like to thank Michael Faux for pointing out an incorrect wording with respect to the definition of central charges.

Author Contributions

Sylvester James Gates, Jr. provided the original idea for the paper the the template for all calculations. Vincent G. J. Rodgers, Leo Rodriguez, and Kory Stiffler wrote the *Mathematica* code to find the transformation laws and algebra. James Parker completed calculations by hand to double check the *Mathematica* code. Kory Stiffler completed the calculations of the Lagrangian symmetries by hand. Sylvester James Gates, Jr. and Kory Stiffler wrote the text for the paper.

Conflicts of Interest

The authors declare no conflict of interest.

A. Explicit Calculation of Symmetries, Including Redundancies

This section explains in more detail the procedure which led us to the symmetries presented in the body of the paper. Many symmetries found in this manner are redundant, and those presented in the paper are the unique symmetries found through this procedure.

A1. $\mathcal{N} = 4$ SUSY-YM: Second Order Bosonic Symmetries

In this section, we explicitly show how the second order bosonic symmetries are discovered through the $\mathcal{N} = 4$ algebra. Several are redundant, and in the body of the paper, only the unique symmetries were listed.

$$\delta^{(2)}_{BS1}(P_1, P_2)A^K \equiv [\delta^{(1)}_{BS1}(P_1), \delta^{(1)}_{BS1}(P_2)]A^K = \Lambda^{KJ}_{1,1}(P_1, P_2)\Box A^J \tag{A1}$$
$$\delta^{(2)}_{BS1}(T_1, T_2)A^K \equiv [\delta^{(1)}_{BS3}(T_1), \delta^{(1)}_{BS3}(T_2)]A^K = \Lambda^{JK}_{3,3}(T_1, T_2)\Box A^J \tag{A2}$$
$$\delta^{(2)}_{BS2}(Q_1, Q_2)B^K \equiv [\delta^{(1)}_{BS2}(Q_1), \delta^{(1)}_{BS2}(Q_2)]B^K = \Lambda^{KJ}_{2,2}(Q_1, Q_2)\Box B^J \tag{A3}$$
$$\delta^{(2)}_{BS2}(T_1, T_2)B^K \equiv [\delta^{(1)}_{BS4}(T_1), \delta^{(1)}_{BS4}(T_2)]B^K = \Lambda^{JK}_{4,4}(T_1, T_2)\Box B^J \tag{A4}$$

$$\delta^{(2)}_{BS3}(T_1, T_2)F^K \equiv [\delta^{(1)}_{BS3}(T_1), \delta^{(1)}_{BS3}(T_2)]F^K = \Lambda^{JK}_{3,3}(T_1, T_2)\Box F^J \tag{A5}$$
$$\delta^{(2)}_{BS4}(T_1, T_2)G^K \equiv [\delta^{(1)}_{BS4}(T_1), \delta^{(1)}_{BS4}(T_2)]G^K = \Lambda^{JK}_{4,4}(T_1, T_2)\Box G^J \tag{A6}$$
$$\delta^{(2)}_{BS5}(U_1, U_2)F^K \equiv [\delta^{(1)}_{BS7}(U_1), \delta^{(1)}_{BS7}(U_2)]F^K = (\Lambda^{\mu\nu}_{7,7})^{JK}(U_1, U_2)\partial_\mu\partial_\nu F^J \tag{A7}$$
$$\begin{aligned}\delta^{(2)}_{BS6}(U_1, U_2)G^K &\equiv [\delta^{(1)}_{BS8}(U_1), \delta^{(1)}_{BS8}(U_2)]G^K + \eta_{\mu\nu}(\Lambda^{\mu\nu}_{8,8})^{JK}(U_1, U_2)\Box G^K \\ &= (\Lambda^{\mu\nu}_{8,8})^{JK}(U_1, U_2)\partial_\mu\partial_\nu G^J\end{aligned} \tag{A8}$$
$$\delta^{(2)}_{BS7}(U_1, U_2)A_\nu \equiv [\delta^{(1)}_{BS8}(U_1), \delta^{(1)}_{BS8}(U_2)]A_\nu = \eta_{\nu\beta}(\Lambda^{\mu\beta}_{8,8})^{JJ}(U_1, U_2)\partial^\alpha F_{\mu\alpha} \tag{A9}$$

with

$$\begin{aligned}\Lambda^{KJ}_{1,1}(P_1, P_2) &\equiv P^K_{[1}P^J_{2]}, \\ \Lambda^{KJ}_{3,3}(T_1, T_2) = \Lambda^{KJ}_{4,4}(T_1, T_2) &\equiv T^{KM}_{[1}T^{MJ}_{2]}, \\ \Lambda^{KJ}_{2,2}(Q_1, Q_2) &\equiv Q^K_{[1}Q^J_{2]}, \\ (\Lambda^{\mu\nu}_{7,7})^{JK}(U_1, U_2) = (\Lambda^{\mu\nu}_{8,8})^{JK}(U_1, U_2) &\equiv (U^\mu_{[1]})^J(U^\nu_{2]})^K,\end{aligned} \tag{A10}$$

and

$$
\begin{aligned}
\delta^{(2)}_{BS8}(P,Q)\begin{pmatrix} A^K \\ B^K \end{pmatrix} \equiv& [\delta^{(1)}_{BS1}(P), \delta^{(1)}_{BS2}(Q)]\begin{pmatrix} A^K \\ B^K \end{pmatrix} \\
=& \Lambda^{IJ}_{1,2}(P,Q)\begin{pmatrix} -\delta^{IK}\Box B^J \\ \delta^{JK}\Box A^I \end{pmatrix} \\
\delta^{(2)}_{BS9}(P,U)\begin{pmatrix} A^K \\ F^K \end{pmatrix} \equiv& [\delta^{(1)}_{BS1}(P), \delta^{(1)}_{BS7}(U)]\begin{pmatrix} A^K \\ F^K \end{pmatrix} \\
=& -(\Lambda^{\mu}_{1,7})^{IJ}(P,U)\begin{pmatrix} \delta^{IK}\partial_\mu F^J \\ \delta^{JK}\partial_\mu\Box A^I \end{pmatrix} \\
\delta^{(2)}_{BS12}(T,U)\begin{pmatrix} A^K \\ \mathrm{d} \end{pmatrix} \equiv& [\delta^{(1)}_{BS3}(T), \delta^{(1)}_{BS8}(U)]\begin{pmatrix} A^K \\ \mathrm{d} \end{pmatrix} \\
=& -(\Lambda^{\mu}_{3,8})^{K}(T,U)\begin{pmatrix} \partial_\mu \mathrm{d} \\ \partial_\mu\Box A^K \end{pmatrix} \\
\delta^{(2)}_{BS14}(T,U)\begin{pmatrix} B^K \\ A_\nu \end{pmatrix} \equiv& [\delta^{(1)}_{BS4}(T), \delta^{(1)}_{BS8}(U)]\begin{pmatrix} B^K \\ A_\nu \end{pmatrix} \\
=& -(\Lambda^{\mu}_{4,8})^{K}(T,U)\begin{pmatrix} \partial^\nu F_{\mu\nu} \\ \eta_{\mu\nu}\Box B^K \end{pmatrix} \\
\delta^{(2)}_{BS10}(Q,U)\begin{pmatrix} B^K \\ F^K \end{pmatrix} \equiv& [\delta^{(1)}_{BS2}(Q), \delta^{(1)}_{BS7}(U)]\begin{pmatrix} B^K \\ F^K \end{pmatrix} \\
=& (\Lambda^{\mu}_{2,7})^{IJ}(Q,U)\begin{pmatrix} \delta^{IK}\partial_\mu F^J \\ \delta^{KJ}\partial_\mu\Box B^I \end{pmatrix} \\
\delta^{(2)}_{BS15}(P,T)\begin{pmatrix} F^K \\ \mathrm{d} \end{pmatrix} \equiv& [\delta^{(1)}_{BS1}(P), \delta^{(1)}_{BS3}(T)]\begin{pmatrix} F^K \\ \mathrm{d} \end{pmatrix} \\
=& \Lambda^{K}_{1,3}(P,T)\begin{pmatrix} -\Box \mathrm{d} \\ \Box F^K \end{pmatrix} \\
\delta^{(2)}_{BS16}(Q,T)\begin{pmatrix} G^K \\ \mathrm{d} \end{pmatrix} \equiv& [\delta^{(1)}_{BS2}(Q), \delta^{(1)}_{BS4}(T)]\begin{pmatrix} G^K \\ \mathrm{d} \end{pmatrix} \\
=& \Lambda^{K}_{2,4}(Q,T)\begin{pmatrix} \Box \mathrm{d} \\ -\Box G^K \end{pmatrix}
\end{aligned}
\tag{A11}
$$

with

$$\begin{aligned}
\Lambda^{JK}_{1,2}(P,Q) &= -\Lambda^{KJ}_{2,1}(Q,P) \equiv P^J Q^K, \\
(\Lambda^{\mu}_{1,7})^{JK}(P,U) &= -(\Lambda^{\mu}_{7,1})^{KJ}(U,P) \equiv P^J (U^\mu)^K, \\
(\Lambda^{\mu}_{3,7})^{K}(T,U) &= -(\Lambda^{\mu}_{5,2})^{K}(U,T) \\
&= (\Lambda^{\mu}_{4,12})^{K}(T,U) = -(\Lambda^{\mu}_{8,4})^{K}(U,T) \equiv T^{KM}(U^\mu)^M, \\
(\Lambda^{\mu}_{2,7})^{JK}(Q,U) &= -(\Lambda^{\mu}_{7,2})^{KJ}(U,Q) \equiv Q^J (U^\mu)^K, \\
\Lambda^{K}_{1,3}(P,T) &= -\Lambda^{K}_{3,1}(T,P) \equiv P^M T^{MK}, \\
\Lambda^{K}_{2,4}(Q,T) &= -\Lambda^{K}_{4,2}(T,Q) \\
&= \Lambda^{K}_{11,14}(Q,T) = -\Lambda^{K}_{14,11}(T,Q) \equiv Q^M T^{MK},
\end{aligned} \tag{A12}$$

and

$$\begin{aligned}
\delta^{(2)}_{BS30}(Q,W)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \equiv& [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS12}(W)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \\
=& \Lambda^{K}_{9,12}(Q,W)(\gamma^5)_c{}^d\begin{pmatrix}\Box\psi^K_c\\ \Box\lambda_d\end{pmatrix} \\
\delta^{(2)}_{BS32}(Q,V)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \equiv& [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS13}(V)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \\
=& (\Lambda^{\alpha}_{9,13})^K(Q,V)\begin{pmatrix}(\gamma^5\gamma^\mu\gamma_\alpha\gamma^\nu)_c{}^d\\ \partial_\mu\partial_\nu\psi^K_d(\gamma^5\gamma_\alpha)_c{}^d\Box\lambda_d\end{pmatrix} \\
\delta^{(2)}_{BS31}(Q,T)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \equiv& [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS14}(T)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \\
=& \Lambda^{K}_{9,14}(Q,T)\begin{pmatrix}-\Box\psi^K_c\\ \Box\lambda_c\end{pmatrix} \\
\delta^{(2)}_{BS26}(W,V)\psi^K_c \equiv& [\delta^{(1)}_{BS12}(W),\delta^{(1)}_{BS13}(V)]\psi^K_c \\
=& (\Lambda^{\mu}_{12,13})^{[JK]}(\gamma_\mu)_c{}^d\Box\psi^J_d + \\
&+ 2(\Lambda^{\mu}_{12,13})^{KJ}(\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^J_d \\
\delta^{(2)}_{BS28}(W,T)\psi^K_c \equiv& [\delta^{(1)}_{BS12}(W),\delta^{(1)}_{BS14}(T)]\psi^K_c \\
=& -\Lambda^{KJ}_{12,14}(W,T)(\gamma^5)_c{}^d\Box\psi^J_d \\
\delta^{(2)}_{BS23}(V,T)\psi^K_c \equiv& [\delta^{(1)}_{BS13}(V),\delta^{(1)}_{BS14}(T)]\psi^K_c + \delta^{(2)}_{BS25}(V,T)\psi^K_c \\
=& (\Lambda^{\alpha}_{13,14})^{[JK]}(V,T)(\gamma^5\gamma_\alpha)_c{}^d\Box\psi^J_d \\
\delta^{(2)}_{BS20}(Q_1,Q_2)\psi^K_c \equiv& [\delta^{(1)}_{BS9}(Q_1),\delta^{(1)}_{BS9}(Q_2)]\psi^K_c \\
=& -\Lambda^{JK}_{11,11}(Q_1,Q_2)\Box\psi^J_c \\
\delta^{(2)}_{BS20}(W_1,W_2)\psi^K_c \equiv& [\delta^{(1)}_{BS12}(W_1),\delta^{(1)}_{BS12}(W_2)]\psi^K_c \\
=& \Lambda^{KJ}_{12,12}(W_1,W_2)\Box\psi^J_c \\
\delta^{(2)}_{BS21}(V_1,V_2)\psi^K_c \equiv& [\delta^{(1)}_{BS13}(V_1),\delta^{(1)}_{BS13}(V_2)]\psi^K_c \\
=& -(\Lambda^{\rho\sigma}_{13,13})^{KJ}(V_1,V_2)(\gamma_\rho\gamma^\mu\gamma_\sigma\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^J_d \\
\delta^{(2)}_{BS20}(T_1,T_2)\psi^K_c \equiv& [\delta^{(1)}_{BS10}(T_1),\delta^{(1)}_{BS10}(T_2)]\psi^K_c \\
=& -\Lambda^{KJ}_{14,14}(T_1,T_2)\Box\psi^J_c
\end{aligned} \tag{A13}$$

with

$$\begin{aligned}
\Lambda^{JK}_{9,9}(Q_1,Q_2) &\equiv Q^J_{[1}Q^K_{2]},\\
\Lambda^{K}_{9,12}(Q,W) &= -\Lambda^{K}_{12,11}(W,Q) \equiv Q^M W^{MK},\\
(\Lambda^{\mu}_{9,13})^{K}(Q,V) &= -\Lambda^{K}_{13,11}(V,Q) \equiv Q^M (V^{\mu})^{MK},\\
\Lambda^{JK}_{12,12}(W_1,W_2) &\equiv W^{JM}_{[1}W^{MK}_{2]},\\
(\Lambda^{\mu}_{12,13})^{JK}(W,V) &= -(\Lambda^{\mu}_{13,12})^{KJ}(V,W) \equiv W^{JM}(V^{\mu})^{MK},\\
\Lambda^{JK}_{12,14}(W,T) &= -\Lambda^{KJ}_{14,12}(T,W) \equiv W^{M(J}T^{K)M},\\
(\Lambda^{\rho\sigma}_{13,13})^{KJ}(V_1,V_2) &\equiv (V^{\rho}_{[1})^{KM}(V^{\sigma}_{2]})^{MJ},\\
(\Lambda^{W}_{13,14})^{JK}(V,T) &= -(\Lambda^{W}_{14,13})^{KJ}(T,V) \equiv (V^{W})^{JM}T^{MK},\\
\Lambda^{KJ}_{14,14}(T_1,T_2) &\equiv T^{KM}_{[1}T^{MJ}_{2]}
\end{aligned} \tag{A14}$$

and

$$\begin{aligned}
\delta^{(2)}_{BS1}(W_1,W_2)A^J \equiv& [\delta^{(1)}_{BS5}(W_1),\delta^{(1)}_{BS5}(W_2)]A^J = \Lambda^{IJ}_{5,5}(W_1,W_2)\Box A^I\\
\delta^{(2)}_{BS4}(W_1,W_2)G^J \equiv& [\delta^{(1)}_{BS5}(W_1),\delta^{(1)}_{BS5}(W_2)]G^J = \Lambda^{IJ}_{5,5}(W_1,W_2)\Box G^I\\
\delta^{(2)}_{BS5}(V_1,V_2)F^J \equiv& [\delta^{(1)}_{BS6}(V_1),\delta^{(1)}_{BS6}(V_2)]F^J = -(\Lambda^{\mu\nu}_{6,6})^{IJ}(V_1,V_2)\partial_\mu\partial_\nu F^I\\
\delta^{(2)}_{BS6}(V_1,V_2)G^J \equiv& [\delta^{(1)}_{BS6}(V_1),\delta^{(1)}_{BS6}(V_2)]G^J = -(\Lambda^{\mu\nu}_{6,6})^{IJ}(V_1,V_2)\partial_\mu\partial_\nu G^I
\end{aligned} \tag{A15}$$

$$
\begin{aligned}
\delta^{(2)}_{BS16}(P,W)\begin{pmatrix} G^J \\ \mathrm{d} \end{pmatrix} \equiv& [\delta^{(1)}_{BS1}(P),\delta^{(1)}_{BS5}(W)]\begin{pmatrix} G^J \\ \mathrm{d} \end{pmatrix} \\
=&\Lambda^{J}_{1,5}(P,W)\begin{pmatrix} -\Box \mathrm{d} \\ \Box G^J \end{pmatrix} \\
\delta^{(2)}_{BS18}(T,W)\begin{pmatrix} F^J \\ G^J \end{pmatrix} \equiv& [\delta^{(1)}_{BS3}(T),\delta^{(1)}_{BS5}(W)]\begin{pmatrix} F^J \\ G^J \end{pmatrix} \\
=&\Lambda^{IK}_{3,5}(T,W)\begin{pmatrix} -\delta^{IJ}\Box G^K \\ \Box\delta^{JK}F^I \end{pmatrix} \\
\delta^{(2)}_{BS8}(T,W)\begin{pmatrix} A^J \\ B^J \end{pmatrix} \equiv& [\delta^{(1)}_{BS4}(T),\delta^{(1)}_{BS5}(W)]\begin{pmatrix} A^J \\ B^J \end{pmatrix} \\
=&\Lambda^{IK}_{4,5}(T,W)\begin{pmatrix} \delta^{JK}\Box B^I \\ -\delta^{IJ}\Box A^K \end{pmatrix} \\
\delta^{(2)}_{BS13}(U,W)\begin{pmatrix} A^J \\ A_\nu \end{pmatrix} \equiv& [\delta^{(1)}_{BS8}(U),\delta^{(1)}_{BS5}(W)]\begin{pmatrix} A^J \\ A_\nu \end{pmatrix} \\
=&-(\Lambda^{\mu}_{8,5})^J(U,W)\begin{pmatrix} \partial^\nu F_{\mu\nu} \\ \eta_{\mu\nu}\Box A^J \end{pmatrix} \\
\delta^{(2)}_{BS9}(W,V)\begin{pmatrix} A^J \\ F^J \end{pmatrix} \equiv& -[\delta^{(1)}_{BS5}(W),\delta^{(1)}_{BS6}(V)]\begin{pmatrix} A^J \\ F^J \end{pmatrix} \\
=&-(\Lambda^{\mu}_{5,6})^{IK}(W,V)\begin{pmatrix} \delta^{IJ}\partial_\mu F^K \\ \delta^{JK}\partial_\mu\Box A^I \end{pmatrix} \\
\delta^{(2)}_{BS11}(T,V)\begin{pmatrix} A^J \\ G^J \end{pmatrix} \equiv& -[\delta^{(1)}_{BS3}(T),\delta^{(1)}_{BS6}(V)]\begin{pmatrix} A^J \\ G^J \end{pmatrix} \\
=&(\Lambda^{\mu}_{3,6})^{IK}(T,V)\begin{pmatrix} \delta^{IJ}\partial_\mu G^K \\ \delta^{JK}\partial_\mu\Box A^I \end{pmatrix} \\
\delta^{(2)}_{BS10}(T,V)\begin{pmatrix} B^J \\ F^J \end{pmatrix} \equiv& -[\delta^{(1)}_{BS4}(T),\delta^{(1)}_{BS6}(V)]\begin{pmatrix} B^J \\ F^J \end{pmatrix} \\
=&-\Lambda^{\mu}_{4,6})^{IK}(T,V)\begin{pmatrix} \delta^{IJ}(\partial_\mu F^K \\ \delta^{JK}\partial_\mu\Box B^I \end{pmatrix} \\
\delta^{(2)}_{BS17}(U,V)\begin{pmatrix} G^J \\ \mathrm{d} \end{pmatrix} \equiv& -[\delta^{(1)}_{BS7}(U),\delta^{(1)}_{BS6}(V)]\begin{pmatrix} G^J \\ \mathrm{d} \end{pmatrix} \\
=&(\Lambda^{\mu\nu}_{7,6})^J(U,V)\begin{pmatrix} -\partial_\mu\partial_\nu \mathrm{d} \\ \partial_\mu\partial_\nu G^J \end{pmatrix} \\
\delta^{(2)}_{BS19}(U,V)\begin{pmatrix} F^J \\ A_\alpha \end{pmatrix} \equiv& -[\delta^{(1)}_{BS8}(U),\delta^{(1)}_{BS6}(V)]\begin{pmatrix} F^J \\ A_\alpha \end{pmatrix} \\
=&(\Lambda^{\mu\nu}_{8,6})^J(U,V)\begin{pmatrix} \partial_\nu\partial^\alpha F_{\mu\alpha} \\ -\eta_{\mu\alpha}\partial_\nu F^J \end{pmatrix}
\end{aligned} \tag{A16}
$$

with

$$\begin{aligned}
\Lambda^{IJ}_{5,5}(W_1,W_2) &\equiv W^{KI}_{[1}W^{JK}_{2]},\\
\Lambda^{IJ}_{6,6}(V_1,V_2) &\equiv (V^{\mu}_{[1})^{KI}(V^{\nu}_{2]})^{JK},\\
\Lambda^{J}_{1,5}(P,W) &\equiv P^K W^{KJ},\\
\Lambda^{IJ}_{3,5}(T,W) &= \Lambda^{IJ}_{4,5}(T,W) \equiv T^{IK}W^{KJ},\\
(\Lambda^{\mu}_{8,5})^J(U,W) &\equiv (U^\mu)^K W^{KJ},\\
(\Lambda^{\mu}_{5,6})^{JI}(W,V) &\equiv W^{JK}(V^\mu)^{KI},\\
(\Lambda^{\mu}_{3,6})^{JI}(T,V) &= (\Lambda^{\mu}_{4,6})^{JI}(T,V) \equiv T^{JK}(V^\mu)^{KI},\\
(\Lambda^{\mu\nu}_{7,6})^{J}(U,V) &= (\Lambda^{\mu\nu}_{8,6})^{J}(U,V) \equiv (U^\mu)^K(V^\nu)^{KJ}
\end{aligned} \tag{A17}$$

and

$$\begin{aligned}
\delta^{(2)}_{BS24}(Q,U)\lambda_c \equiv& [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS10}(U)]\lambda_c\\
=& -2(\Lambda^{\mu}_{9,10})^{KK}(Q,U)(\gamma^5\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\lambda_d,\\
\delta^{(2)}_{BS25}(Q,U)\psi^K_c \equiv& [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS10}(U)]\psi^K_c\\
=& -2(\Lambda^{\mu}_{9,10})^{KJ}(Q,U)(\gamma^5\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\psi^J_d,\\
\delta^{(2)}_{BS34}(W,U)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \equiv& [\delta^{(1)}_{BS12}(W),\delta^{(1)}_{BS10}(U)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix}\\
=& -(\Lambda^{\mu}_{12,10})^{K}(W,U)\begin{pmatrix}(\gamma_\mu)_c{}^d\Box\psi^K_d\\ (\gamma^\nu\gamma_\mu\gamma^\alpha)_c{}^d\partial_\nu\partial_\alpha\lambda_d\end{pmatrix}\\
\delta^{(2)}_{BS36}(V,U)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \equiv& [\delta^{(1)}_{BS13}(V),\delta^{(1)}_{BS10}(U)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix}\\
=& (\Lambda^{\mu\nu}_{13,10})^{K}(V,U)\begin{pmatrix}-(\gamma_\nu\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\psi^K_d\\ (\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\lambda_d\end{pmatrix}\\
\delta^{(2)}_{BS33}(T,U)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \equiv& [\delta^{(1)}_{BS14}(T),\delta^{(1)}_{BS10}(U)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix}\\
=& -(\Lambda^{\mu}_{14,10})^{K}(T,U)\begin{pmatrix}(\gamma^5\gamma_\mu)_c{}^d\Box\psi^K_d\\ (\gamma^5\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\lambda_d\end{pmatrix}\\
\delta^{(2)}_{BS22}(U_1,U_2)\lambda_c \equiv& [\delta^{(1)}_{BS10}(U_1),\delta^{(1)}_{BS10}(U_2)]\lambda_c\\
=& (\Lambda^{\mu\nu}_{10,10})^{KK}(U_1,U_2)(\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\lambda_d\\
\delta^{(2)}_{BS21}(U_1,U_2)\psi^K_c \equiv& [\delta^{(1)}_{BS10}(U_1),\delta^{(1)}_{BS10}(U_2)]\psi^K_c\\
=& (\Lambda^{\mu\nu}_{10,10})^{KJ}(U_1,U_2)(\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\psi^J_d\\
\delta^{(2)}_{BS27}(U,P)\lambda_c \equiv& [\delta^{(1)}_{BS10}(U),\delta^{(1)}_{BS11}(P)]\lambda_c = 2(\Lambda^{\mu}_{10,11})^{KK}(U,P)(\gamma^\nu)_c{}^d\partial_\mu\partial_\nu\lambda_d\\
\delta^{(2)}_{BS26}(U,P)\psi^K_c \equiv& [\delta^{(1)}_{BS10}(U),\delta^{(1)}_{BS11}(P)]\psi^K_c\\
=& (\Lambda^{\mu}_{10,11})^{JK}(U,P)(\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\psi^J_d\\
&+(\Lambda^{\mu}_{10,11})^{KJ}(U,P)(\gamma_\mu)_c{}^d\Box\psi^J_d\\
\delta^{(2)}_{BS29}(Q,P)\lambda_c \equiv& [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS11}(P)]\lambda_c = -\Lambda^{KK}_{9,11}(Q,P)(\gamma^5)_c{}^d\Box\lambda_d
\end{aligned} \tag{A18}$$

$$
\begin{aligned}
\delta^{(2)}_{BS28}(Q,P)\psi^K_c &\equiv [\delta^{(1)}_{BS9}(Q),\delta^{(1)}_{BS11}(P)]\psi^K_c = -\Lambda^{KJ}_{9,11}(Q,P)(\gamma^5)_c{}^d\Box\psi^J_d \\
\delta^{(2)}_{BS31}(W,P)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} &\equiv [\delta^{(1)}_{BS12}(W),\delta^{(1)}_{BS11}(P)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \\
&= \Lambda^K_{9,11}(W,P)\begin{pmatrix}\Box\psi^K_c\\ -\Box\lambda_d\end{pmatrix} \\
\delta^{(2)}_{BS35}(V,P)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} &\equiv [\delta^{(1)}_{BS13}(V),\delta^{(1)}_{BS11}(P)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \\
&= (\Lambda^\mu_{13,11})^K(V,P)\begin{pmatrix}(\gamma^\alpha\gamma_\mu\gamma^\beta)_c{}^d\partial_\alpha\partial_\beta\psi^K_d\\ (\gamma_\mu)_c{}^d\Box\lambda_d\end{pmatrix} \\
\delta^{(2)}_{BS30}(T,P)\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} &\equiv [\delta^{(1)}_{BS14}(T),\delta^{(1)}_{BS11}(P)]\begin{pmatrix}\lambda_c\\ \psi^K_c\end{pmatrix} \\
&= -(\Lambda^\mu_{14,11})^K(T,P)\begin{pmatrix}(\gamma^5)_c{}^d\Box\psi^K_d\\ (\gamma^5)_c{}^d\Box\lambda_d\end{pmatrix} \\
\delta^{(2)}_{BS20}(P_1,P_2)\psi^K_c &\equiv [\delta^{(1)}_{BS11}(P_1),\delta^{(1)}_{BS11}(P_2)]\psi^K_c \\
&= -\Lambda^{KJ}_{11,11}(P_1,P_2)\Box\psi^J_c,
\end{aligned} \tag{A19}
$$

with

$$
\begin{aligned}
(\Lambda^\mu_{9,10})^{JK}(Q,U) &\equiv Q^J(U^\mu)^K,\quad (\Lambda^\mu_{12,10})^K(W,U) = W^{KM}(U^\mu)^M, \\
(\Lambda^{\mu\nu}_{13,10})^K(V,U) &\equiv (V^\mu)^{KM}(U^\nu)^M,\quad (\Lambda^\mu_{14,10})^K(T,U) = T^{KM}(U^\mu)^M, \\
(\Lambda^{\mu\nu}_{10,10})^{KJ}(U_1,U_2) &\equiv (U^\mu_{[1})^K(U^\nu_{2]})^J,\quad (\Lambda^\mu_{10,11})^{KM}(U,P) \equiv (U^\mu)^K P^M, \\
\Lambda^{KJ}_{9,11}(Q,P) &\equiv Q^{(K}P^{J)},\quad \Lambda^K_{12,11}(W,P) \equiv W^{KM}P^M, \\
(\Lambda^\mu_{13,11})^K(V,P) &\equiv (V^\mu)^{KM}P^{M)},\quad \Lambda^K_{14,11}(T,P) \equiv T^{KM}P^M, \\
\Lambda^{KM}_{11,11}(P_1,P_2) &\equiv P^K_{[1}P^M_{2]}
\end{aligned} \tag{A20}
$$

where [ ] denotes antisymmetry, *i.e.*, $U^J_{[1}U^K_{2]} = U^J_1 U^K_2 - U^J_2 U^K_1$.

A2. $\mathcal{N}=4$ SUSY-YM: Fermionic Symmetries

Taking the commutators or D_a and D^I_a with the first order bosonic symmetries for the $\mathcal{N}=4$ SUSY-YM system, we find several first order *fermionic* symmetries, some of which are redundant. The symmetries calculated below which involve $\varepsilon^a_I P^K\epsilon^{IJK}$, $\varepsilon^a_I Q^K\epsilon^{IJK}$, and $\varepsilon^a_I(U^\rho)^K\epsilon^{IJK}$ are redefined through

$$
\begin{aligned}
\varepsilon^a_I P^K\epsilon^{IJK} &\to \varepsilon^a P^J \\
\varepsilon^a_I Q^K\epsilon^{IJK} &\to \varepsilon^a Q^J \\
\varepsilon^a_I(U^\rho)^K\epsilon^{IJK} &\to \varepsilon^a(U^\rho)^J
\end{aligned} \tag{A21}
$$

as symmetries defined either way are equivalent for the Lagrangian. In Section 2.4.3, all symmetries are listed using this redefinition where applicable.

$$\begin{aligned}
\delta^{(1)}_{FS19}(P)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv & \varepsilon^a P^J \begin{pmatrix} \Box\psi^J_a \\ i(\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} = -\varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS1}(P)] \begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \\
\delta^{(1)}_{FS13}(Q)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv & \varepsilon^a Q^J \begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^J_b \\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} = \varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS2}(Q)] \begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \\
\delta^{(1)}_{FS54}(U)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv & \varepsilon^a (U^\mu)^J \partial_\mu \begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\nu\psi^J_b \\ iC_{ab}\mathrm{d} \end{pmatrix} \\
= & \varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS7}(U)] \begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \\
\delta^{(1)}_{FS70}(U)\begin{pmatrix} A_\mu \\ \psi^J_b \end{pmatrix} \equiv & \varepsilon^a (U_\mu)^J \partial^\nu \begin{pmatrix} i(\gamma^5\gamma_\nu)_a{}^b\psi^J_b \\ -(\gamma^5)_{ab}F^\mu{}_\nu \end{pmatrix} \\
= & \varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS8}(U)] \begin{pmatrix} A_\mu \\ \psi^J_b \end{pmatrix} \\
\delta^{(1)}_{FS2}(P)\begin{pmatrix} A^J \\ \lambda_b \end{pmatrix} \equiv & \varepsilon^a P^J \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box A^J \end{pmatrix} = \varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS1}(P)] \begin{pmatrix} A^J \\ \lambda_b \end{pmatrix} \\
\delta^{(1)}_{FS3}(Q)\begin{pmatrix} B^J \\ \lambda_b \end{pmatrix} \equiv & \varepsilon^a Q^J \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box B^J \end{pmatrix} \\
= & -\varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS2}(Q)] \begin{pmatrix} B^J \\ \lambda_b \end{pmatrix} \\
\delta^{(1)}_{FS51}(U)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \equiv & \varepsilon^a (U^\mu)^J \begin{pmatrix} \partial^\nu\partial_{[\mu}(\gamma_{\nu]})_a{}^b\lambda_b \\ (\sigma^\nu{}_\mu)_{ab}\partial_\nu G^J \end{pmatrix} \\
= & \varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS8}(U)] \begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \\
\delta^{(1)}_{FS46}(U)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} \equiv & \varepsilon^a (U^\mu)^J \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ i(\gamma^5)_{ab}\partial_\mu F^J \end{pmatrix} \\
= & -i\varepsilon^a [\mathrm{D}_a, \delta^{(1)}_{BS7}(U)] \begin{pmatrix} F^J \\ \lambda_b \end{pmatrix}
\end{aligned} \tag{A22}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS3}(T)]$ we have

$$\begin{aligned}
\delta^{(1)}_{FS21}(T)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv & \varepsilon^a T^{JM} \begin{pmatrix} (\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ iC_{ab}\Box A^M \end{pmatrix} \\
\delta^{(1)}_{FS82}(T)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv & \varepsilon^a T^{JM} \begin{pmatrix} \Box\psi^M_a \\ -i(\gamma^\mu)_{ab}\partial_\mu F^M \end{pmatrix}
\end{aligned} \tag{A23}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS4}(T)]$

$$\delta^{(1)}_{FS28}(T)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} (\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box B^M \end{pmatrix}$$
$$\delta^{(1)}_{FS35}(T)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^M_b \\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{A24}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS9}(Q)]$

$$\delta^{(1)}_{FS66}(Q)\begin{pmatrix} A_\mu \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a Q^J\begin{pmatrix} -(\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\psi^J_b \\ \frac{1}{2}(\gamma^\alpha\sigma^{\mu\nu})_{ba}\partial_\alpha F_{\mu\nu} \end{pmatrix}$$
$$\delta^{(1)}_{FS13}(Q)\begin{pmatrix} d \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a Q^J\begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^J_b \\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu d \end{pmatrix} \tag{A25}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS12}(W)]$

$$\delta^{(1)}_{FS27}(W)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} (\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box A^M \end{pmatrix}$$
$$\delta^{(1)}_{FS22}(W)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} (\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ C_{ab}\Box B^M \end{pmatrix}$$
$$\delta^{(1)}_{FS36}(W)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^M_b \\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu F^M \end{pmatrix}$$
$$\delta^{(1)}_{FS83}(W)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} i\Box\psi^M_a \\ (\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{A26}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS13}(V)]$

$$.\delta^{(1)}_{FS76}(V)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM}\begin{pmatrix} (\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5\gamma_\rho)_{ab}\Box A^M \end{pmatrix}$$
$$\delta^{(1)}_{FS81}(V)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM}\begin{pmatrix} -i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ (\gamma_\rho)_{ab}\Box B^M \end{pmatrix}$$
$$\delta^{(1)}_{FS43}(V)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM}\begin{pmatrix} (\gamma^5\gamma^\nu\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^M \end{pmatrix}$$
$$\delta^{(1)}_{FS56}(V)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\rho)^{JM}\begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ -(\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^M \end{pmatrix} \tag{A27}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS14}(T)]$

$$\delta^{(1)}_{FS21}(T)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} (\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ iC_{ab}\Box A^M \end{pmatrix}$$
$$\delta^{(1)}_{FS28}(T)\begin{pmatrix} B^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} (\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box B^M \end{pmatrix} \tag{A28}$$

$$\delta^{(1)}_{FS82}(T)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} i\Box\psi^M_a \\ (\gamma^\mu)_{ab}\partial_\mu F^M \end{pmatrix}$$
$$\delta^{(1)}_{FS35}(T)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a T^{JM}\begin{pmatrix} i(\gamma^5)_a{}^b\Box\psi^M_b \\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{A29}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS5}(W)]$

$$\delta^{(1)}_{FS27}(W)\begin{pmatrix} A^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} (\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box A^M \end{pmatrix}$$
$$\delta^{(1)}_{FS83}(W)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a W^{JM}\begin{pmatrix} i\Box\psi^M_a \\ (\gamma^\mu)_{ab}\partial_\mu G^M \end{pmatrix} \tag{A30}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS6}(V)]$

$$\delta^{(1)}_{FS42}(V)\begin{pmatrix} F^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\mu)^{JM}\begin{pmatrix} i(\gamma^5\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ (\gamma^5)_{ab}\partial_\mu F^M \end{pmatrix}$$
$$\delta^{(1)}_{FS57}(V)\begin{pmatrix} G^J \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a (V^\mu)^{JM}\begin{pmatrix} (\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\psi^M_b \\ -iC_{ab}\partial_\mu G^M \end{pmatrix} \tag{A31}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS9}(Q)]$

$$\delta^{(1)}_{FS7}(Q)\begin{pmatrix} A^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a Q^J\begin{pmatrix} (\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ -iC_{ab}\Box A^J \end{pmatrix}$$
$$\delta^{(1)}_{FS3}(Q)\begin{pmatrix} B^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a Q^J\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box B^J \end{pmatrix}$$
$$\delta^{(1)}_{FS16}(Q)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a Q^J\begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix}$$
$$\delta^{(1)}_{FS11}(Q)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a Q^J\begin{pmatrix} i(\gamma^5)_a{}^b\Box\lambda_b \\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \tag{A32}$$

and and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS1}(P)]$

$$\delta^{(1)}_{FS1}(P)\begin{pmatrix} A^K \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I P^K\begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^I_b \\ \delta^{IJ}(\gamma^5)_{ab}\Box A^K \end{pmatrix}$$
$$\delta^{(1)}_{FS20}(P)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I P^I\begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu\mathrm{d} \end{pmatrix} \tag{A33}$$

$$\delta^{(1)}_{FS19}(P)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I P^K\epsilon^{IJK}\begin{pmatrix} \Box\psi^J_b \\ i(\gamma^\mu)_{ab}\partial_\mu\mathrm{d} \end{pmatrix}$$
$$\rightarrow \varepsilon^a P^J\begin{pmatrix} \Box\psi^J_b \\ i(\gamma^\mu)_{ab}\partial_\mu\mathrm{d} \end{pmatrix} \tag{A34}$$

and from $[\mathrm{D}_a^I, \delta_{BS3}^{(1)}(T)]$

$$\delta_{FS24}^{(1)}(T)\begin{pmatrix} A^M \\ \psi_b^J \end{pmatrix} \equiv \varepsilon_I^a \epsilon^{IJK} T^{KM} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \psi_b^J \\ C_{ab} \Box A^M \end{pmatrix}$$
$$\delta_{FS86}^{(1)}(T)\begin{pmatrix} F^M \\ \psi_b^J \end{pmatrix} \equiv \varepsilon_I^a \epsilon^{IJK} T^{KM} \begin{pmatrix} \Box \psi_a^J \\ i(\gamma^\mu)_{ab} \partial_\mu F^M \end{pmatrix}$$
$$\delta_{FS23}^{(1)}(T)\begin{pmatrix} A^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon_I^a T^{IK} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ C_{ab} \Box A^K \end{pmatrix}$$
$$\delta_{FS84}^{(1)}(T)\begin{pmatrix} F^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon_I^a T^{IK} \begin{pmatrix} \Box \lambda_a \\ i(\gamma^\mu)_{ab} \partial_\mu F^K \end{pmatrix} \tag{A35}$$

and from $[\mathrm{D}_a^I, \delta_{BS2}^{(1)}(Q)]$

$$\delta_{FS4}^{(1)}(Q)\begin{pmatrix} B^K \\ \psi_b^J \end{pmatrix} \equiv \varepsilon_I^a Q^K \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi_b^I \\ \delta^{IJ}(\gamma^5)_{ab} \Box B^K \end{pmatrix}$$
$$\delta_{FS13}^{(1)}(P)\begin{pmatrix} \mathrm{d} \\ \psi_b^K \end{pmatrix} \equiv \varepsilon_I^a Q^J \epsilon^{IJK} \begin{pmatrix} i(\gamma^5)_a{}^b \Box \psi_b^K \\ -(\gamma^5\gamma^\mu)_{ab} \partial_\mu \mathrm{d} \end{pmatrix}$$
$$\to \varepsilon^a Q^K \begin{pmatrix} i(\gamma^5)_a{}^b \Box \psi_b^K \\ -(\gamma^5\gamma^\mu)_{ab} \partial_\mu \mathrm{d} \end{pmatrix}$$
$$\delta_{FS14}^{(1)}(Q)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon_I^a Q^I \begin{pmatrix} i(\gamma^5)_a{}^b \Box \lambda_b \\ -(\gamma^5\gamma^\mu)_{ab} \partial_\mu \mathrm{d} \end{pmatrix} \tag{A36}$$

and from $[\mathrm{D}_a^I, \delta_{BS4}^{(1)}(T)]$

$$\delta_{FS34}^{(1)}(T)\begin{pmatrix} B^M \\ \psi_b^J \end{pmatrix} \equiv \varepsilon_I^a \epsilon^{IJK} T^{KM} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi_b^J \\ (\gamma^5)_{ab} \Box B^M \end{pmatrix}$$
$$\delta_{FS39}^{(1)}(T)\begin{pmatrix} G^K \\ \psi_b^J \end{pmatrix} \equiv \varepsilon_I^a \epsilon^{IJK} T^{KM} \begin{pmatrix} (\gamma^5)_a{}^b \Box \psi_b^N \\ i(\gamma^5\gamma^\mu)_{ab} \partial_\mu G^M \end{pmatrix}$$
$$\delta_{FS32}^{(1)}(T)\begin{pmatrix} B^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon_I^a T^{IK} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ (\gamma^5)_{ab} \Box B^K \end{pmatrix}$$
$$\delta_{FS38}^{(1)}(T)\begin{pmatrix} G^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon_I^a T^{IK} \begin{pmatrix} (\gamma^5)_a{}^b \Box \lambda_b \\ i(\gamma^5\gamma^\mu)_{ab} \partial_\mu G^K \end{pmatrix} \tag{A37}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS7}(U)]$

$$\delta^{(1)}_{FS47}(U)\begin{pmatrix} F^K \\ \psi^I_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^K \partial_\mu \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b \partial_\nu \psi^I_b \\ i(\gamma^5)_{ab} F^K \end{pmatrix}$$

$$\delta^{(1)}_{FS54}(U)\begin{pmatrix} \mathrm{d} \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^K \epsilon^{IJK} \begin{pmatrix} (\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^J_b \\ iC_{ab}\partial_\mu \mathrm{d} \end{pmatrix}$$

$$\to \varepsilon^a (U^\mu)^J \begin{pmatrix} (\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^J_b \\ iC_{ab}\partial_\mu \mathrm{d} \end{pmatrix}$$

$$\delta^{(1)}_{FS55}(U)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^I \partial_\mu \begin{pmatrix} (\gamma^\nu)_a{}^b \partial_\nu \lambda_b \\ iC_{ab} \mathrm{d} \end{pmatrix} \quad \text{(A38)}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS8}(U)]$

$$\delta^{(1)}_{FS52}(U)\begin{pmatrix} G^K \\ \psi^I_b \end{pmatrix} \equiv \varepsilon^a_I (U^\mu)^K \begin{pmatrix} \partial^\nu \partial_{[\mu}(\gamma_{\nu]})_a{}^b \psi^I_b \\ (\sigma^\nu{}_\mu)_{ab}\partial_\nu G^K \end{pmatrix}$$

$$\delta^{(1)}_{FS70}(U)\begin{pmatrix} A_\nu \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I (U_\nu)^K \epsilon^{IJK} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi^J_b \\ -(\gamma^5)_{ab}\partial^\mu F^\nu{}_\mu \end{pmatrix}$$

$$\to \varepsilon^a (U_\nu)^J \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi^J_b \\ -(\gamma^5)_{ab}\partial^\mu F^\nu{}_\mu \end{pmatrix}$$

$$\delta^{(1)}_{FS71}(U)\begin{pmatrix} A_\nu \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U_\nu)^I \begin{pmatrix} -i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ (\gamma^5)_{ab}\partial^\mu F^\nu{}_\mu \end{pmatrix} \quad \text{(A39)}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS9}(Q)]$

$$\delta^{(1)}_{FS8}(Q)\begin{pmatrix} A^I \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_I Q^K \begin{pmatrix} (\gamma^\mu)_a{}^b \partial_\mu \psi^K_b \\ -iC_{ab}\Box A^I \end{pmatrix}$$

$$\delta^{(1)}_{FS4}(Q)\begin{pmatrix} B^I \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_I Q^K \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi^K_b \\ (\gamma^5)_{ab}\Box B^I \end{pmatrix}$$

$$\delta^{(1)}_{FS15}(Q)\begin{pmatrix} F^I \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_I Q^K \begin{pmatrix} \Box \psi^K_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^I \end{pmatrix}$$

$$\delta^{(1)}_{FS12}(Q)\begin{pmatrix} G^I \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a_I Q^K \begin{pmatrix} (\gamma^5)_a{}^b \Box \psi^K_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^I \end{pmatrix} \quad \text{(A40)}$$

and from $[\mathrm{D}_a^I, \delta_{BS9}^{(1)}(Q)]$

$$\begin{aligned}
\delta_{FS67}^{(1)}(Q)\begin{pmatrix} A_\nu \\ \lambda_b \end{pmatrix} \equiv& \varepsilon_I^a Q^I \begin{pmatrix} -(\gamma_\nu\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ \frac{1}{2}(\gamma^\alpha\sigma^{\mu\nu})_{ba}\partial_\alpha F_{\mu\nu} \end{pmatrix} \\
\delta_{FS14}^{(1)}(Q)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv& \varepsilon_I^a Q^I \begin{pmatrix} i(\gamma^5)_a{}^b\Box\lambda_b \\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \\
\delta_{FS7}^{(1)}(Q)\begin{pmatrix} A^J \\ \lambda_b \end{pmatrix} \equiv& \varepsilon_I^a Q^K \epsilon^{IJK} \begin{pmatrix} (\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ -iC_{ab}\Box A^J \end{pmatrix} \\
\to& \varepsilon^a Q^J \begin{pmatrix} (\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ -iC_{ab}\Box A^J \end{pmatrix} \\
\delta_{FS3}^{(1)}(Q)\begin{pmatrix} B^J \\ \lambda_b \end{pmatrix} \equiv& \varepsilon_I^a Q^K \epsilon^{IJK} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box B^J \end{pmatrix} \\
\to& \varepsilon^a Q^J \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box B^J \end{pmatrix} \\
\delta_{FS16}^{(1)}(Q)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} \equiv& \varepsilon_I^a Q^K \epsilon^{IJK} \begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \\
\to& \varepsilon^a Q^J \begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \\
\delta_{FS11}^{(1)}(Q)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \equiv& \varepsilon_I^a Q^K \epsilon^{IJK} \begin{pmatrix} (\gamma^5)_a{}^b\Box\lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \\
\to& \varepsilon^a Q^J \begin{pmatrix} (\gamma^5)_a{}^b\Box\lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix}
\end{aligned} \tag{A41}$$

and from $[\mathrm{D}_a^I, \delta_{BS12}^{(1)}(W)]$

$$\begin{aligned}
.\delta_{FS30}^{(1)}(W)\begin{pmatrix} A^J \\ \psi_b^M \end{pmatrix} \equiv& \ \varepsilon_I^a \epsilon^{IJK} W^{KM} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi_b^M \\ (\gamma^5)_{ab}\Box A^J \end{pmatrix} \\
\delta_{FS26}^{(1)}(W)\begin{pmatrix} B^J \\ \psi_b^M \end{pmatrix} \equiv& \ \varepsilon_I^a \epsilon^{IJK} W^{KM} \begin{pmatrix} i(\gamma^\mu)_a{}^b\partial_\mu\psi_b^M \\ C_{ab}\Box B^J \end{pmatrix} \\
\delta_{FS37}^{(1)}(W)\begin{pmatrix} F^J \\ \psi_b^M \end{pmatrix} \equiv& \ \varepsilon_I^a \epsilon^{IJK} W^{KM} \begin{pmatrix} (\gamma^5)_a{}^b\Box\psi_b^M \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \\
\delta_{FS88}^{(1)}(W)\begin{pmatrix} G^J \\ \psi_b^M \end{pmatrix} \equiv& \ \varepsilon_I^a \epsilon^{IJK} W^{KM} \begin{pmatrix} \Box\psi_a^M \\ i(\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \\
\delta_{FS69}^{(1)}(W)\begin{pmatrix} A_\nu \\ \psi_b^M \end{pmatrix} \equiv& \ \varepsilon_I^a W^{IM} \begin{pmatrix} -(\gamma^5\gamma_\nu\gamma^\mu)_a{}^b\partial_\mu\psi_b^M \\ -\frac{1}{2}(\gamma^5\gamma^\mu\sigma^{\alpha\nu})_{ba}\partial_\mu F_{\alpha\nu} \end{pmatrix} \\
\delta_{FS90}^{(1)}(W)\begin{pmatrix} \mathrm{d} \\ \psi_b^M \end{pmatrix} \equiv& \ \varepsilon_I^a W^{IM} \begin{pmatrix} \Box\psi_a^M \\ i(\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix}
\end{aligned} \tag{A42}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS13}(V)$

$$\begin{aligned}
\delta^{(1)}_{FS75}(V)\begin{pmatrix} A^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} -(\gamma^5\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ i(\gamma^5\gamma_\rho)_{ab} \Box A^J \end{pmatrix} \\
\delta^{(1)}_{FS79}(V)\begin{pmatrix} B^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ (\gamma_\rho)_{ab} \Box B^J \end{pmatrix} \\
\delta^{(1)}_{FS48}(V)\begin{pmatrix} F^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} (\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu \partial_\nu \psi^M_b \\ -i(\gamma^5\gamma_\rho\gamma^\mu)_{ba} \partial_\mu F^J \end{pmatrix} \\
\delta^{(1)}_{FS61}(V)\begin{pmatrix} G^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} (V^\rho)^{KM} \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu \partial_\nu \psi^M_b \\ (\gamma_\rho\gamma^\mu)_{ba} \partial_\mu G^J \end{pmatrix} \\
\delta^{(1)}_{FS73}(V)\begin{pmatrix} A_\nu \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I (V^\rho)^{IM} \begin{pmatrix} (\gamma^5\gamma_\nu\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ \frac{1}{2}(\gamma^5\gamma_\rho\gamma^\mu\sigma^{\alpha\nu})_{ba} \partial_\mu F_{\alpha\nu} \end{pmatrix} \\
\delta^{(1)}_{FS63}(V)\begin{pmatrix} \mathrm{d} \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I (V^\rho)^{IM} \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu \partial_\nu \psi^M_b \\ (\gamma_\rho\gamma^\mu)_{ba} \partial_\mu \mathrm{d} \end{pmatrix}
\end{aligned} \tag{A43}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS14}(T)]$

$$\begin{aligned}
\delta^{(1)}_{FS25}(T)\begin{pmatrix} A^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} i(\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ C_{ab} \Box A^J \end{pmatrix} \\
\delta^{(1)}_{FS33}(T)\begin{pmatrix} B^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ (\gamma^5)_{ab} \Box B^J \end{pmatrix} \\
\delta^{(1)}_{FS85}(T)\begin{pmatrix} F^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} \Box \psi^M_a \\ i(\gamma^\mu)_{ab} \partial_\mu F^J \end{pmatrix} \\
\delta^{(1)}_{FS40}(T)\begin{pmatrix} G^J \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I \epsilon^{IJK} T^{KM} \begin{pmatrix} (\gamma^5)_a{}^b \Box \psi^M_b \\ i(\gamma^5\gamma^\mu)_{ab} \partial_\mu G^J \end{pmatrix} \\
\delta^{(1)}_{FS68}(T)\begin{pmatrix} A_\nu \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I T^{IM} \begin{pmatrix} (\gamma_\nu\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ -\frac{1}{2}(\gamma^\mu\sigma^{\alpha\nu})_{ba} \partial_\mu F_{\alpha\nu} \end{pmatrix} \\
\delta^{(1)}_{FS41}(T)\begin{pmatrix} \mathrm{d} \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I T^{IM} \begin{pmatrix} (\gamma^5)_a{}^b \Box \psi^M_b \\ i(\gamma^5\gamma^\mu)_{ab} \partial_\mu \mathrm{d} \end{pmatrix}
\end{aligned} \tag{A44}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS5}(W)]$

$$\begin{aligned}
\delta^{(1)}_{FS31}(W)\begin{pmatrix} A^M \\ \psi^J_b \end{pmatrix} &\equiv \varepsilon^a_I W^{KM} \epsilon^{IJK} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \psi^J_b \\ (\gamma^5)_{ab} \Box A^M \end{pmatrix} \\
\delta^{(1)}_{FS89}(W)\begin{pmatrix} G^M \\ \psi^J_b \end{pmatrix} &\equiv \varepsilon^a_I W^{KM} \epsilon^{IJK} \begin{pmatrix} \Box \psi^N_a \\ i(\gamma^\mu)_{ab} \partial_\mu G^M \end{pmatrix} \\
\delta^{(1)}_{FS29}(W)\begin{pmatrix} A^M \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I W^{IM} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ (\gamma^5)_{ab} \Box A^M \end{pmatrix} \\
\delta^{(1)}_{FS87}(W)\begin{pmatrix} G^M \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I W^{IM} \begin{pmatrix} \Box \lambda_a \\ i(\gamma^\mu)_{ab} \partial_\mu G^M \end{pmatrix}
\end{aligned} \tag{A45}$$

and from $[\mathrm{D}_a^I, \delta^{(1)}_{BS6}(V)]$

$$\delta^{(1)}_{FS45}(V)\begin{pmatrix} F^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I (V^\mu)^{KM} \epsilon^{IJK} \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^J_b \\ i(\gamma^5)_{ab}\partial_\mu F^M \end{pmatrix}$$
$$\delta^{(1)}_{FS60}(V)\begin{pmatrix} G^M \\ \psi^J_b \end{pmatrix} \equiv \varepsilon^a_I (V^\mu)^{KM} \epsilon^{IJK} \begin{pmatrix} (\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^N_a \\ iC_{ab}\partial_\mu G^M \end{pmatrix}$$
$$\delta^{(1)}_{FS44}(V)\begin{pmatrix} F^M \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (V^\mu)^{IM} \begin{pmatrix} (\gamma^5\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \lambda_b \\ i(\gamma^5)_{ab}\partial_\mu F^M \end{pmatrix}$$
$$\delta^{(1)}_{FS59}(V)\begin{pmatrix} G^M \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (V^\mu)^{IM} \begin{pmatrix} (\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \lambda_b \\ iC_{ab}\partial_\mu G^M \end{pmatrix} \tag{A46}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS10}(U)]$

$$\delta^{(1)}_{FS63}(U)\begin{pmatrix} \mathrm{d} \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^K_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu \mathrm{d} \end{pmatrix}$$
$$\delta^{(1)}_{FS73}(U)\begin{pmatrix} A_\mu \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} (\gamma^5\gamma_\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\nu \psi^K_b \\ \frac{1}{2}(\gamma^5\gamma_\rho\gamma^\mu\sigma^{\alpha\beta})_{ba}\partial_\mu F_{\alpha\beta} \end{pmatrix} \tag{A47}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS10}(U)]$

$$\delta^{(1)}_{FS77}(U)\begin{pmatrix} A^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} (\gamma^5\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ -i(\gamma^5\gamma_\rho)_{ab}\Box A^K \end{pmatrix}$$
$$\delta^{(1)}_{FS78}(U)\begin{pmatrix} B^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \lambda_b \\ (\gamma_\rho)_{ab}\Box B^K \end{pmatrix}$$
$$\delta^{(1)}_{FS49}(U)\begin{pmatrix} F^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} (\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \lambda_b \\ -i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^K \end{pmatrix}$$
$$\delta^{(1)}_{FS53}(U)\begin{pmatrix} G^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a (U^\rho)^K \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \lambda_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^K \end{pmatrix} \tag{A48}$$

and from $[\mathrm{D}_a^I, \delta^{(1)}_{BS10}(U)]$

$$\delta^{(1)}_{FS74}(U)\begin{pmatrix} A^I \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^M \begin{pmatrix} -(\gamma^5\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ i(\gamma^5\gamma_\rho)_{ab}\Box A^I \end{pmatrix}$$
$$\delta^{(1)}_{FS80}(U)\begin{pmatrix} B^I \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^M \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b \partial_\mu \psi^M_b \\ (\gamma_\rho)_{ab}\Box B^I \end{pmatrix}$$
$$\delta^{(1)}_{FS50}(U)\begin{pmatrix} F^I \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^M \begin{pmatrix} (\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^M_b \\ -i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^I \end{pmatrix}$$
$$\delta^{(1)}_{FS58}(U)\begin{pmatrix} G^I \\ \psi^M_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^M \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b \partial_\mu\partial_\nu \psi^M_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^I \end{pmatrix} \tag{A49}$$

and from $[\mathrm{D}_a^I, \delta^{(1)}_{BS10}(U)]$

$$\delta^{(1)}_{FS77}(U)\begin{pmatrix} A^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^K \epsilon^{IJK}\begin{pmatrix} (\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ -i(\gamma^5\gamma_\rho)_{ab}\Box A^J \end{pmatrix}$$
$$\to \varepsilon^a (U^\rho)^J \begin{pmatrix} (\gamma^5\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ -i(\gamma^5\gamma_\rho)_{ab}\Box A^J \end{pmatrix}$$
$$\delta^{(1)}_{FS78}(U)\begin{pmatrix} B^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^K \epsilon^{IJK}\begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma_\rho)_{ab}\Box B^J \end{pmatrix}$$
$$\to \varepsilon^a (U^\rho)^J \begin{pmatrix} i(\gamma_\rho\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma_\rho)_{ab}\Box B^J \end{pmatrix}$$
$$\delta^{(1)}_{FS49}(U)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^K \epsilon^{IJK}\begin{pmatrix} -(\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^J \end{pmatrix}$$
$$\to \varepsilon^a (U^\rho)^J \begin{pmatrix} -(\gamma^5\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ i(\gamma^5\gamma_\rho\gamma^\mu)_{ba}\partial_\mu F^J \end{pmatrix}$$
$$\delta^{(1)}_{FS53}(U)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} \varepsilon^a_I (U^\rho)^K \epsilon^{IJK}\begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^J \end{pmatrix}$$
$$\to \varepsilon^a (U^\rho)^J \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu G^J \end{pmatrix}$$
$$\delta^{(1)}_{FS62}(U)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^I \begin{pmatrix} i(\gamma^\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\mu\partial_\nu\lambda_b \\ (\gamma_\rho\gamma^\mu)_{ba}\partial_\mu \mathrm{d} \end{pmatrix}$$
$$\delta^{(1)}_{FS72}(U)\begin{pmatrix} A_\mu \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a_I (U^\rho)^I \begin{pmatrix} (\gamma^5\gamma_\mu\gamma_\rho\gamma^\nu)_a{}^b\partial_\nu\lambda_b \\ \frac{1}{2}(\gamma^5\gamma_\rho\gamma^\nu\sigma^{\alpha\beta})_{ba}\partial_\nu F_{\alpha\beta} \end{pmatrix} \quad \text{(A50)}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS11}(P)]$

$$\delta^{(1)}_{FS19}(U)\begin{pmatrix} \mathrm{d} \\ \psi^K_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} i\Box\psi^k_b \\ -(\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix}$$
$$\delta^{(1)}_{FS65}(U)\begin{pmatrix} A_\mu \\ \psi^I_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} (\gamma^5\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\psi^K_b \\ -i(\gamma^5\gamma^\nu)_{ab}\partial^\mu F_{\mu\nu} \end{pmatrix} \quad \text{(A51)}$$

and from $[\mathrm{D}_a, \delta^{(1)}_{BS11}(P)]$

$$\delta^{(1)}_{FS2}(P)\begin{pmatrix} A^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box A^K \end{pmatrix}$$
$$\delta^{(1)}_{FS5}(P)\begin{pmatrix} B^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} i(\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ C_{ab}\Box B^K \end{pmatrix}$$
$$\delta^{(1)}_{FS9}(P)\begin{pmatrix} F^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} (\gamma^5)_a{}^b\Box\lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^K \end{pmatrix}$$
$$\delta^{(1)}_{FS17}(P)\begin{pmatrix} G^K \\ \lambda_b \end{pmatrix} \equiv \varepsilon^a P^K \begin{pmatrix} \Box\lambda_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^K \end{pmatrix} \quad \text{(A52)}$$

and from $[\mathrm{D}^I_a, \delta^{(1)}_{BS11}(P)]$

$$\begin{aligned}
\delta^{(1)}_{FS1}(P)\begin{pmatrix} A^I \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I P^M \begin{pmatrix} -(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ i(\gamma^5)_{ab}\Box A^I \end{pmatrix} \\
\delta^{(1)}_{FS6}(P)\begin{pmatrix} B^I \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I P^M \begin{pmatrix} i(\gamma^\mu)_a{}^b\partial_\mu\psi^M_b \\ C_{ab}\Box B^I \end{pmatrix} \\
\delta^{(1)}_{FS10}(P)\begin{pmatrix} F^I \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I P^M \begin{pmatrix} (\gamma^5)_a{}^b\Box\psi^M_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^I \end{pmatrix} \\
\delta^{(1)}_{FS18}(P)\begin{pmatrix} G^I \\ \psi^M_b \end{pmatrix} &\equiv \varepsilon^a_I P^M \begin{pmatrix} \Box\psi^M_a \\ i(\gamma^\mu)_{ab}\partial_\mu G^I \end{pmatrix}
\end{aligned} \tag{A53}$$

from $[\mathrm{D}^I_a, \delta^{(1)}_{BS11}(P)]$

$$\begin{aligned}
\delta^{(1)}_{FS20}(P)\begin{pmatrix} \mathrm{d} \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I P^I \begin{pmatrix} -i\Box\lambda_b \\ (\gamma^\mu)_{ab}\partial_\mu \mathrm{d} \end{pmatrix} \\
\delta^{(1)}_{FS64}(P)\begin{pmatrix} A_\mu \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I P^I \begin{pmatrix} (\gamma^5\gamma_\mu\gamma^\nu)_a{}^b\partial_\nu\lambda_b \\ -i(\gamma^5\gamma^\nu)_{ab}\partial^\mu F_{\mu\nu} \end{pmatrix} \\
\delta^{(1)}_{FS2}(P)\begin{pmatrix} A^J \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I P^K \epsilon^{IJK} \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box A^J \end{pmatrix} \\
&\to \varepsilon^a P^J \begin{pmatrix} i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ (\gamma^5)_{ab}\Box A^J \end{pmatrix} \\
\delta^{(1)}_{FS5}(P)\begin{pmatrix} B^J \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I P^K \epsilon^{IJK} \begin{pmatrix} i(\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ C_{ab}\Box B^J \end{pmatrix} \\
&\to \varepsilon^a P^J \begin{pmatrix} i(\gamma^\mu)_a{}^b\partial_\mu\lambda_b \\ C_{ab}\Box B^J \end{pmatrix} \\
\delta^{(1)}_{FS9}(P)\begin{pmatrix} F^J \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I P^K \epsilon^{IJK} \begin{pmatrix} (\gamma^5)_a{}^b\Box\lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \\
&\to \varepsilon^a P^J \begin{pmatrix} (\gamma^5)_a{}^b\Box\lambda_b \\ i(\gamma^5\gamma^\mu)_{ab}\partial_\mu F^J \end{pmatrix} \\
\delta^{(1)}_{FS17}(P)\begin{pmatrix} G^J \\ \lambda_b \end{pmatrix} &\equiv \varepsilon^a_I P^K \epsilon^{IJK} \begin{pmatrix} -i\Box\lambda_a \\ (\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix} \\
&\to \varepsilon^a P^J \begin{pmatrix} -i\Box\lambda_a \\ (\gamma^\mu)_{ab}\partial_\mu G^J \end{pmatrix}
\end{aligned} \tag{A54}$$

A3. $\mathcal{N} = 2$ FH

In this section, we list all of the $\mathcal{N} = 2$ FH fermionic first order symmetries uncovered via our method, including the redundant ones. Only the unique symmetries were listed in the body of the paper. From from $[\tilde{\mathrm{D}}^i_a, \tilde{\delta}^{(1)}_{BS1}(\tilde{T})]$ we find the symmetries

$$\begin{aligned}
\tilde{\delta}^{(1)}_{FS1}(\tilde{T})\begin{pmatrix}\tilde{A}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i\tilde{T}^{ik}\begin{pmatrix}-(\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^1_b\\ iC_{ab}\Box\tilde{A}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS2}(\tilde{T})\begin{pmatrix}\tilde{F}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i\tilde{T}^{ik}\begin{pmatrix}\Box\tilde{\psi}^1_a\\ i(\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS3}(\tilde{T})\begin{pmatrix}\tilde{A}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^2)^{ij}\tilde{T}^{jk}\begin{pmatrix}i(\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^2_b\\ C_{ab}\Box\tilde{A}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS4}(\tilde{T})\begin{pmatrix}\tilde{F}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^2)^{ij}T^{jk}\begin{pmatrix}-i\Box\tilde{\psi}^2_a\\ (\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k\end{pmatrix}
\end{aligned} \tag{A55}$$

and from $[\tilde{\mathrm{D}}^i_a, \tilde{\delta}^{(1)}_{BS2}(\tilde{T})]$

$$\begin{aligned}
\tilde{\delta}^{(1)}_{FS5}(\tilde{T})\begin{pmatrix}\tilde{B}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^3)^{ij}\tilde{T}^{jk}\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^1_b\\ (\gamma^5)_{ab}\Box\tilde{B}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS6}(\tilde{T})\begin{pmatrix}\tilde{G}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^3)^{ij}\tilde{T}^{jk}\begin{pmatrix}-i(\gamma^5)_a{}^b\Box\tilde{\psi}^1_b\\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS7}(\tilde{T})\begin{pmatrix}\tilde{B}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^1)^{ij}\tilde{T}^{jk}\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^2_b\\ (\gamma^5)_{ab}\Box\tilde{B}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS8}(\tilde{T})\begin{pmatrix}\tilde{G}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^1)^{ij}\tilde{T}^{jk}\begin{pmatrix}-i(\gamma^5)_a{}^b\Box\tilde{\psi}^2_b\\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k\end{pmatrix}
\end{aligned} \tag{A56}$$

Calculation of $[\tilde{\mathrm{D}}^i_a, \tilde{\delta}^{(1)}_{BS3}(\tilde{T})]$ uncovers no new symmetries, just these same eight again:

$$\begin{aligned}
\tilde{\delta}^{(1)}_{FS1}(\tilde{T})\begin{pmatrix}\tilde{A}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^2)^{ik}\tilde{T}^{12}\begin{pmatrix}i(\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^1_b\\ C_{ab}\Box\tilde{A}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS2}(\tilde{T})\begin{pmatrix}\tilde{F}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^2)^{ik}\tilde{T}^{12}\begin{pmatrix}i\Box\tilde{\psi}^1_a\\ -(\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS3}(\tilde{T})\begin{pmatrix}\tilde{A}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_k\tilde{T}^{12}\begin{pmatrix}-(\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^2_b\\ iC_{ab}\Box\tilde{A}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS4}(\tilde{T})\begin{pmatrix}\tilde{F}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_k\tilde{T}^{12}\begin{pmatrix}\Box\tilde{\psi}^2_a\\ i(\gamma^\mu)_{ab}\partial_\mu\tilde{F}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS5}(\tilde{T})\begin{pmatrix}\tilde{B}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^1)^{ik}\tilde{T}^{12}\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^1_b\\ (\gamma^5)_{ab}\Box\tilde{B}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS6}(\tilde{T})\begin{pmatrix}\tilde{G}^k\\ \tilde{\psi}^1_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^1)^{ik}\tilde{T}^{12}\begin{pmatrix}i(\gamma^5)_a{}^b\Box\tilde{\psi}^1_b\\ -(\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS7}(\tilde{T})\begin{pmatrix}\tilde{B}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^3)^{ik}\tilde{T}^{12}\begin{pmatrix}i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\tilde{\psi}^2_b\\ (\gamma^5)_{ab}\Box\tilde{B}^k\end{pmatrix}\\
\tilde{\delta}^{(1)}_{FS8}(\tilde{T})\begin{pmatrix}\tilde{G}^k\\ \tilde{\psi}^2_b\end{pmatrix} &\equiv \varepsilon^a_i(\sigma^3)^{ik}\tilde{T}^{12}\begin{pmatrix}-i(\gamma^5)_a{}^b\Box\tilde{\psi}^2_b\\ (\gamma^5\gamma^\mu)_{ab}\partial_\mu\tilde{G}^k\end{pmatrix}
\end{aligned} \tag{A57}$$

under redefinitions of $\tilde{T}$.

References

1. Maldacena, J.M. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **1998**, *2*, 231–252.
2. Klebanov, I.R.; Strassler, M.J. Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities. *JHEP* **2000**, *8*, doi:10.1088/1126-6708/2000/08/052.
3. Maldacena, J.M.; Nunez, C. Towards the large N limit of pure N = 1 super Yang Mills. *Phys. Rev. Lett.* **2001**, *86*, 588–591.
4. Cvetic, M.; Gibbons, G.W.; Lu, H.; Pope, C.N. Supersymmetric non-singular fractional D2-branes and NS-NS 2-branes. *Nucl. Phys.* **2001**, *B606*, 18–44.
5. Maldacena, J.M.; Nastase, H.S. The supergravity dual of a theory with dynamical supersymmetry breaking. *JHEP* **2001**, *9*, doi:10.1088/1126-6708/2001/09/024.
6. Canoura, F.; Edelstein, J.D.; Zayas, L.A.P.; Ramallo, A.V.; Vaman, D. Supersymmetric branes on $AdS_5 \times Y^{p,q}$ and their field theory duals. *JHEP* **2006**, *3*, doi:10.1088/1126-6708/2006/03/101.
7. Herzog, C.P.; Klebanov, I.R. Gravity duals of fractional branes in various dimensions. *Phys. Rev. D* **2001**, *63*, doi:10.1103/PhysRevD.63.126005.
8. Herzog, C.P. String tensions and three dimensional confining gauge theories. *Phys. Rev. D* **2002**, *66*, doi:10.1103/PhysRevD.66.065009.
9. Pando Zayas, L.A.; Sonnenschein, J.; Vaman, D. Regge trajectories revisited in the gauge/string correspondence. *Nucl. Phys.* **2004**, *B682*, 3–44.
10. Pando Zayas, L.A.; Rodgers, V.G.J.; Stiffler, K. Luscher term for k-string potential from holographic one loop corrections. *JHEP* **2008**, *12*, doi:10.1088/1126-6708/2008/12/036.
11. Doran, C.A.; Pando Zayas, L.A.; Rodgers, V.G.J.; Stiffler, K. Tensions and Luscher terms for (2+1)-dimensional k-strings from holographic models. *JHEP* **2009**, *11*, doi:10.1088/1126-6708/2009/11/064.
12. Stiffler, K. Mesons From string theory. *ArXiv E-Prints*, **2009**, arXiv:0909.5681. Available online: http://arxiv.org/abs/0909.5681 (accessed on 16 June 2015).
13. Stiffler, K. A Walk through Superstring Theory with an Application to Yang-Mills Theory: K-Strings and D-branes as Gauge/gravity Dual Objects. Ph.D. Thesis, University of Iowa, Iowa City, IA, USA, 30 July 2010.
14. Siegel, W.; Rocek, M. On off-shell supermultiplets. *Phys. Lett. B* **1981**, *105*, 275–277.
15. Aharony, O.; Bergman, O.; Jafferis, D.L.; Maldacena, J. N = 6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals. *JHEP* **2008**, *10*, doi:10.1088/1126-6708/2008/10/091.
16. Gates, S.J., Jr.; Nishino, H. Remarks on the N = 2 supersymmetric Chern-Simons theories. *Phys. Lett. B* **1992**, *281*, 72–80.
17. Nishino, H.; Gates, S.J., Jr. Chern-Simons theories with supersymmetries in three-dimensions. *Int. J. Mod. Phys.* **1993**, *A8*, 3371–3422.
18. Brooks, R.; Gates, S.J., Jr. Extended supersymmetry and superBF gauge theories. *Nucl. Phys.* **1994**, *B432*, 205–224.

19. Gates, S.J., Jr.; Rana, L. A Theory of spinning particles for large N extended supersymmetry. *Phys. Lett. B* **1995**, *352*, 50–58.
20. Galperin, A.; Ivanov, E.; Kalitsyn, S.; Ogievetsky, V.; Sokatchev, E. Unconstrained N = 2 Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace. *Class. Quant. Grav.* **1984**, *1*, 469–498.
21. Galperin, A.; Ivanov, E.; Ogievetsky, V.; Sokatchev, E. Harmonic Supergraphs. Feynman Rules and Examples. *Class. Quant. Grav.* **1985**, *2*, doi:10.1088/0264-9381/2/5/005.
22. Gates, S.J., Jr.; Hull, C.; Rocek, M. Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models. *Nucl. Phys.* **1984**, *B248*, 157–186.
23. Karlhede, A.; Lindstrom, U.; Rocek, M. Selfinteracting tensor multiplets in N = 2 superspace. *Phys. Lett. B* **1984**, *147*, 297–300.
24. Fayet, P. Fermi-Bose hypersymmetry. *Nucl. Phys.* **1976**, *B113*, 135–155.
25. Gates, S.J.J.; Gonzales, J.; MacGregor, B.; Parker, J.; Polo-Sherk, R.; Rodgers, V.G.J.; Wassink, L. 4D, N = 1 Supersymmetry Genomics (I). *JHEP* **2009**, *12*, doi:10.1088/1126-6708/2009/12/008.
26. Polyakov, A.M. Quantum geometry of bosonic strings. *Phys. Lett. B* **1981**, *103*, 207–210.
27. Polyakov, A.M. Quantum geometry of fermionic strings. *Phys. Lett. B* **1981**, *103*, 211–213.