Twisted Sequences of Extensions

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Abstract. Gabber and Joseph [GJ, §5] introduced a ladder diagram between two natural sequences of extensions. Their diagram is used to produce a 'twisted' sequence that is applied to old and new results on extension groups in category $\mathcal{O}$.

1. The Gabber-Joseph Isomorphism

Let $\mathcal{A}$ be an abelian category with enough projectives. Let $\mathrm{E}^p = \mathrm{Ext}^p_{\mathcal{A}}$ (with the convention that $\mathrm{E}^p = 0$ if $p < 0$). Let $\mathrm{H} = \mathrm{E}^0 = \hom_{\mathcal{A}}$. If E is used to represent some E^p, then use the relative notations, E^+ and E^-, to represent E^{p+1} and E^{p-1} respectively.

Suppose that R and T are exact, mutually adjoint endofunctors defined on $\mathcal{A}$. Let $\theta = RT$. The unit of the adjunction (T, R) is $\eta : \mathrm{Id} \to \theta$ and the co-unit of the adjunction (R, T) is $\epsilon : \theta \to \mathrm{Id}$. Use these to define the functors,

$$C = \mathrm{Coker}\,\eta \qquad D = \mathrm{Coim}\,\eta$$
$$K = \mathrm{Ker}\,\epsilon \qquad I = \mathrm{Im}\,\epsilon.$$

There are also natural transformations, $\iota : I \to \mathrm{Id}$ and $\pi : \mathrm{Id} \to D$.

There is a natural adjoint pairing (C, K) so that C is right exact and K is left exact. If M and N are objects in $\mathcal{A}$, there are canonical exact sequences, $KN \hookrightarrow \theta N \twoheadrightarrow IN$ and $DM \hookrightarrow \theta M \twoheadrightarrow CM$. Each gives rise to a long exact sequence of extensions.

Theorem 1.1 [GJ, 5.1.8] *Suppose that M is C-acyclic. There is a natural commutative diagram with exact rows,*

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}(M, KN) & \longrightarrow & \mathrm{E}(M, \theta N) & \longrightarrow & \mathrm{E}(M, IN) & \xrightarrow{\delta_1} & \mathrm{E}^+(M, KN) & \longrightarrow \\
 & \gamma\downarrow & & \beta\downarrow & & \alpha\downarrow & & \gamma^+\downarrow & \\
\longrightarrow & \mathrm{E}(CM, N) & \longrightarrow & \mathrm{E}(\theta M, N) & \longrightarrow & \mathrm{E}(DM, N) & \xrightarrow{\delta_2} & \mathrm{E}^+(CM, N) & \longrightarrow \ ,
\end{array}$$

where β is an isomorphism. If DM is C-acyclic and $IN = N$, then α and γ are isomorphisms.

Proof. Let $P \twoheadrightarrow M$ be a projective resolution. There is an exact sequence of chain complexes,

$$0 \longrightarrow DP \longrightarrow \theta P \longrightarrow CP \longrightarrow 0.$$

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Since M is C-acyclic, this is a resolution of the exact sequence,

$$0 \longrightarrow DM \longrightarrow \theta M \longrightarrow CM \longrightarrow 0. \tag{1.1.1}$$

Let $X \twoheadrightarrow DM$ and $Z \twoheadrightarrow CM$ be projective resolutions. Use the horseshoe lemma [W, 2.28] to construct a split exact sequence resolving diagram (1.1.1),

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0. \tag{1.1.2}$$

By the comparison theorem [W, 2.3.7], there are chain maps $a : X \to DP$ and $c : Z \to CP$ lifting Id_{DM} and Id_{CM} respectively. Using the splitting maps of diagram (1.1.2), construct a chain map $b : Y \to \theta P$ lifting $\mathrm{Id}_{\theta M}$ and completing a commutative diagram of chain complexes with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & DP & \longrightarrow & \theta P & \longrightarrow & CP & \longrightarrow & 0. \end{array}$$

Applying $\mathrm{H}(-, N)$ yields a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}(CP, N) & \longrightarrow & \mathrm{H}(\theta P, N) & \longrightarrow & \mathrm{H}(DP, N) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}(Z, N) & \longrightarrow & \mathrm{H}(Y, N) & \longrightarrow & \mathrm{H}(X, N) & \longrightarrow & 0 \end{array} \tag{1.1.3}$$

Since P is a projective complex, there is also a natural commutative diagram of complexes with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}(P, KN) & \longrightarrow & \mathrm{H}(P, \theta N) & \longrightarrow & \mathrm{H}(P, IN) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & {\scriptstyle \phi_P}\downarrow & & \\ 0 & \longrightarrow & \mathrm{H}(CP, N) & \longrightarrow & \mathrm{H}(\theta P, N) & \longrightarrow & \mathrm{H}(DP, N)\,. & & \end{array} \tag{1.1.4}$$

The chain map ϕ_P is uniquely defined by the diagram because $\mathrm{H}(\pi_P, N)\phi_P = \mathrm{H}(P, \iota_N)$. The first two vertical mappings are isomorphisms.

Combining diagram (1.1.3) and diagram (1.1.4), and applying [W, 1.3.4] yields the Gabber-Joseph diagram. Since $\theta P \twoheadrightarrow \theta M$ is a projective resolution, b is a homotopy equivalence so β is an isomorphism. (So far, this is the same as the proof given in [GJ, 5.1.8].)

Let $f : P \to X$ be a chain map lifting π_M. Then, by the uniqueness part of the comparison theorem, af is homotopic to π_P. So, $\mathrm{H}(f, N)\,\mathrm{H}(a, N)\phi_P$ is homotopic to $\mathrm{H}(\pi_P, N)\phi_P = \mathrm{H}(P, \iota_N)$. Passing to cohomology, $\mathrm{E}(\pi_M, N)\alpha = \mathrm{E}(M, \iota_N)$.

Now suppose that DM is C-acyclic so that $DX \twoheadrightarrow DM$ is a resolution. The chain map $D(f) : DP \to DX$ lifts Id_{DM} so $D(f)a$ is homotopic to π_X. Hence $\mathrm{H}(a, N)\,\mathrm{H}(D(f), N)\phi_X$ is homotopic to $\mathrm{H}(\pi_X, N)\phi_X = \mathrm{H}(X, \iota_N)$.

By functoriality, $\mathrm{H}(f, N)\,\mathrm{H}(X, \iota_N) = \mathrm{H}(P, \iota_N)\,\mathrm{H}(f, IN)$ and, since π is a natural transformation, $\mathrm{H}(\pi_P, N)\,\mathrm{H}(D(f), N) = \mathrm{H}(f, N)\,\mathrm{H}(\pi_X, N)$. Then,

$$\begin{aligned} \mathrm{H}(\pi_P, N)\,\mathrm{H}(D(f), N)\phi_X &= \mathrm{H}(f, N)\,\mathrm{H}(\pi_X, N)\phi_X = \mathrm{H}(f, N)\,\mathrm{H}(X, \iota_N) \\ &= \mathrm{H}(P, \iota_N)\,\mathrm{H}(f, IN) = \mathrm{H}(\pi_P, N)\phi_P\,\mathrm{H}(f, IN)\,. \end{aligned}$$

Since $\mathrm{H}(\pi_P, N)$ is a monomorphism, $\mathrm{H}(D(f), N)\phi_X = \phi_P \mathrm{H}(f, IN)$ which means that $\mathrm{H}(a, N)\phi_P \mathrm{H}(f, IN)$ is homotopic to $\mathrm{H}(X, \iota_N)$. Passing to cohomology yields $\alpha\, \mathrm{E}(\pi_M, IN) = \mathrm{E}(DM, \iota_N)$.

If $IN = N$, $\mathrm{E}(M, \iota_N) = \mathrm{Id}$ and $\mathrm{E}(DM, \iota_N) = \mathrm{Id}$, so that α is an isomorphism. By the long-five lemma, γ is also an isomorphism. □

Corollary 1.2 *If M and DM are C-acyclic, then* $\mathrm{E}(M, KN)$ *and* $\mathrm{E}(CM, IN)$ *are isomorphic.*

Proof. By standard properties of adjunction maps, $T(\epsilon_N)$ is an epimorphism. So $I(\iota_N)$ is an isomorphism as are $\theta(\iota_N)$ and $K(\iota_N)$. In this way, $I(IN)$, $\theta(IN)$, and $K(IN)$ will be identified with IN, KN, and θN respectively. Applying theorem 1.1 to IN, there is a commutative diagram,

$$\begin{array}{ccc} \mathrm{E}(M, IN) & \xrightarrow{\delta_1} & \mathrm{E}^{+}(M, KN) \\ {\scriptstyle\alpha'}\downarrow & & {\scriptstyle\gamma'}\downarrow \\ \mathrm{E}(DM, IN) & \xrightarrow{\delta_2'} & \mathrm{E}^{+}(CM, IN), \end{array}$$

where the vertical mappings are isomorphisms and the primes indicate maps defined with respect to IN. □

2. The Twisted Sequence

Theorem 2.1 *Suppose that M and DM are C-acyclic. There is a commutative diagram with exact rows,*

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{E}^{-}(DM, JN) & \xrightarrow{\delta} & \mathrm{E}(M, IN) & \xrightarrow{\alpha} & \mathrm{E}(DM, N) & \xrightarrow{\kappa} & \mathrm{E}(DM, JN) & \longrightarrow \\ & \| & & {\scriptstyle\delta_1}\downarrow & & {\scriptstyle\delta_2}\downarrow & & \| & \\ \longrightarrow & \mathrm{E}^{-}(DM, JN) & \xrightarrow{d} & \mathrm{E}^{+}(M, KN) & \xrightarrow{\gamma} & \mathrm{E}^{+}(CM, N) & \xrightarrow{\chi} & \mathrm{E}(DM, JN) & \longrightarrow \ , \end{array}$$

where $JN = \mathrm{Coker}\, \epsilon_N$. If $DM = M$, the first row is the long exact sequence associated to the exact sequence, $IN \hookrightarrow N \twoheadrightarrow JN$.

Proof. Let ϕ'_P be the map defined by (1.1.4) with $N = IN$. Then $\mathrm{H}(\pi_P, IN)\phi'_P = \mathrm{Id}$. Using the notation from the previous section,

$$\begin{aligned} \mathrm{H}(\pi_P, N)\, \mathrm{H}(DP, \iota_N)\phi'_P &= \mathrm{H}(P, \iota_N)\, \mathrm{H}(\pi_P, IN)\phi'_P \\ &= \mathrm{H}(P, \iota_N) = \mathrm{H}(\pi_P, N)\phi_P\, . \end{aligned}$$

Because $\mathrm{H}(\pi_P, N)$ is a monomorphism, $\mathrm{H}(DP, \iota_N)\phi'_P = \phi_P$. Then

$$\mathrm{H}(a, N)\phi_P = \mathrm{H}(a, N)\, \mathrm{H}(DP, \iota_N)\phi'_P = \mathrm{H}(X, \iota_N)\, \mathrm{H}(a, IN)\phi'_P.$$

Taking cohomology, $\alpha = \mathrm{E}(DM, \iota_N)\, \alpha'$. In a similar fashion, $\beta = \mathrm{E}(\theta M, \iota_N)\, \beta'$ and $\gamma = \mathrm{E}(CM, \iota_N)\, \gamma'$.

Diagram 1:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}^{-}(DM,JN) & \xrightarrow{\delta} & \mathrm{E}(M,IN) & \xrightarrow{\alpha} & \mathrm{E}(DM,N) & \xrightarrow{\kappa} & \mathrm{E}(DM,JN) & \longrightarrow \\
& \Vert & & \alpha' \big\downarrow & & \Vert & & \Vert & \\
\longrightarrow & \mathrm{E}^{-}(DM,JN) & \longrightarrow & \mathrm{E}(DM,IN) & \longrightarrow & \mathrm{E}(DM,N) & \longrightarrow & \mathrm{E}(DM,JN) & \longrightarrow
\end{array} \tag{2.1.1}$$

The second row is the long exact sequence associated to $IN \hookrightarrow N \twoheadrightarrow JN$. Since α' is an isomorphism, define δ so that the first square commutes. This produces a commutative diagram with exact rows. If $DM = M$, $\alpha' = \mathrm{Id}$ which proves the second conclusion.

Diagram 2:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}^{-}(DM,JN) & \longrightarrow & \mathrm{E}(DM,IN) & \longrightarrow & \mathrm{E}(DM,N) & \longrightarrow & \mathrm{E}(DM,JN) & \longrightarrow \\
& \delta_3 \big\downarrow & & \delta_2' \big\downarrow & & \delta_2 \big\downarrow & & \delta_3 \big\downarrow & \\
\longrightarrow & \mathrm{E}(CM,JN) & \longrightarrow & \mathrm{E}^{+}(CM,IN) & \longrightarrow & \mathrm{E}^{+}(CM,N) & \longrightarrow & \mathrm{E}^{+}(CM,JN) & \longrightarrow
\end{array} \tag{2.1.2}$$

This is a commutative diagram with exact rows where the vertical maps are the natural connecting maps.

Diagram 3:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}(CM,JN) & \longrightarrow & \mathrm{E}^{+}(CM,IN) & \longrightarrow & \mathrm{E}^{+}(CM,N) & \longrightarrow & \mathrm{E}^{+}(CM,JN) & \longrightarrow \\
& \delta_3 \big\uparrow & & \gamma' \big\uparrow & & \Vert & & \delta_3 \big\uparrow & \\
\longrightarrow & \mathrm{E}^{-}(DM,JN) & \xrightarrow{d} & \mathrm{E}^{+}(M,KN) & \xrightarrow{\gamma} & \mathrm{E}^{+}(CM,N) & \xrightarrow{\chi} & \mathrm{E}(DM,JN) & \longrightarrow
\end{array} \tag{2.1.3}$$

Since $T(\epsilon_N)$ is surjective, $TJN = 0$. By the adjoint pairing (R,T), $\mathrm{E}(\theta M, JN) = \mathrm{E}(TM, TJN) = 0$ so δ_3 is an isomorphism. Define d and χ to make the diagram commutative. Then the second row is also exact.

Assembling the three diagrams proves the first conclusion since $\delta_3^{-1}\,\delta_3 = \mathrm{Id}$ and $(\gamma')^{-1}\,\delta_2'\,\alpha' = \delta_1$. □

The second row of 2.1 will be referred to as a twisted sequence.

3. Applications in category $\mathcal{O}$: older results

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Category $\mathcal{O}$ is the category of $\mathfrak{g}$-modules introduced in [BGG]. For background information on category $\mathcal{O}$, we will rely on [Hum2] where the original sources and the later developments can be found.

Let S be the set of simple root reflections in the Weyl group W. The stabilizer of a weight λ under the dot action is W°_{λ}. Let w_0 denote the longest element and let 1 denote the identity. The Bruhat order on W is denoted by $<$. Let ξ be its characteristic function defined by

$$\xi(x,y)=\begin{cases}1 & \text{if } x\le y \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

Let ℓ denote the length function on W. If $x,y\in W$, $\ell(x,y)=\ell(y)-\ell(x)$.

The R-polynomials are defined in [Hum1, §7]. Let $r_p(x,y)$ denote the coefficient of q^p in $(-1)^{n-p}R_{x,y}$ where $n=\ell(x,y)$. A recursion for $r_p(x,y)$ begins with $r_p(w_0,y)=0$ if $p\neq 0$ and $r_0(w_0,y)=\xi(w_0,y)$. If $x<w_0$, choose an $s\in S$ so that $xs>x$. Then, for all p,

$$r_p(x,y)=\begin{cases}r_p(xs,ys) & \text{if } ys>y,\\ r_p(xs,y)+r_{p-1}(xs,y)-r_{p-1}(xs,ys) & \text{if } ys<y.\end{cases} \tag{3.0.1}$$

The following properties of the r_p can be proved by induction or translated from properties of the R-polynomials in [Hum1, §7]. If $r_p(x,y)\neq 0$, then $x\le y$ and $0\le p\le \ell(x,y)$. Also $r_0=\xi$ and, if $n=\ell(x,y)$, $r_p(x,y)=r_{n-p}(x,y)$.

Specializing (3.0.1) to $p=1$, $r_1(w_0,y)=0$ and, if $xs>x$,

$$r_1(x,y)=\begin{cases}r_1(xs,ys) & \text{if } ys>y,\\ r_1(xs,y)+1 & \text{if } ys<y \text{ and } xs\not< ys,\\ r_1(xs,y) & \text{if } ys<y \text{ and } xs<ys.\end{cases} \tag{3.0.2}$$

Choose anti-dominant integral weights λ and μ so that $W^\circ_\lambda=\{e\}$ and $W^\circ_\mu=\{e,s\}$ where $s\in S$. If $x\in W$, let M_x denote the Verma module with highest weight $x\cdot\lambda$. The block of $\mathcal{O}$ with projective generator M_{w_0} is $\mathcal{O}_\lambda$ [Hum2, 4.9]. Here, T is the translation functor T^μ_λ where R is its left and right adjoint T^λ_μ [Hum2, 7.1-2]. A module $M\in\mathcal{O}_\lambda$ is C-acyclic if, and only if, $DM=M$ [C, 2.9] and this condition is true for each M_x [C, 2.8(i)].

For $x,y\in W$, write $\mathrm{E}^p(x,y)$ for $\mathrm{E}^p(M_x,M_y)$ and $e_p(x,y)$ for its dimension. Also, for x and $z\le y$ in W, write $\mathrm{E}^p(x,y/z)$ for $\mathrm{E}^p(M_x,M_y/M_z)$ and let $e_p(x,y/z)$ be the dimension.

Since M_{w_0} is projective, $e_p(w_0,y)=0$ if $p\neq 0$. By the properties of homomorphisms between Verma modules, $e_0=\xi$ [Hum2, 5.2], so $e_0=r_0$. The vanishing properties also match. If $e_p(x,y)\neq 0$ then $x\le y$ and $0\le p\le\ell(x,y)$ [Hum2, 6.11].

The twisted sequence can be used to re-prove some of the results of [GJ, 5.2].

Proposition 3.1 [GJ, 5.2.1] *Suppose that $xs>x$ and $ys<y$. For all p,*

$$e_p(xs,y)=e_p(x,ys).$$

Proof. Let $M=M_{xs}$ and $N=M_{ys}$. Then $CM=M_x$, $IN=N$, and $KN=M_y$ [C, 3.5]. By 1.2, $\mathrm{E}(xs,y)$ is isomorphic to $\mathrm{E}(x,ys)$. □

Suppose that $xs>x$ and $ys<y$. Apply 2.1 with $M=M_{xs}$ and $N=M_y$. Then $IN=M_{ys}$, $CM=M_x$, and $KN=M_y$. There is a commutative diagram with exact

rows,

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{E}^{-}(xs, y/ys) & \xrightarrow{\delta} & \mathrm{E}(xs, ys) & \xrightarrow{\alpha} & \mathrm{E}(xs, y) & \xrightarrow{\kappa} & \mathrm{E}(xs, y/ys) & \longrightarrow \\ & \| & & \delta_1 \downarrow & & \delta_2 \downarrow & & \| & \\ \longrightarrow & \mathrm{E}^{-}(xs, y/ys) & \xrightarrow{d} & \mathrm{E}^{+}(xs, y) & \xrightarrow{\gamma} & \mathrm{E}^{+}(x, y) & \xrightarrow{\chi} & \mathrm{E}(xs, y/ys) & \longrightarrow \end{array} \quad . \tag{3.1.1}$$

The following result is the twisted equivalent of [GJ, 5.2.3].

Proposition 3.2 *Suppose that $xs > x$ and $ys < y$. For all p,*

$$e_p(x, y) - e_p(xs, y) \geq e_{p-1}(xs, y) - e_{p-1}(xs, ys)$$

and this is an equality if, and only if, $\operatorname{Ker} d^{\,p-1} = \operatorname{Ker} \delta^{\,p-1}$ *and* $\operatorname{Ker} d^{\,p-2} = \operatorname{Ker} \delta^{\,p-2}$.

Proof. Since $d = \delta_1 \delta$, $\operatorname{Ker} \delta \subseteq \operatorname{Ker} d$. Identify E with E^{p-1} and d with $d^{\,p-2}$ in diagram (3.1.1). Because the second row is exact, there is a short exact sequence

$$0 \longrightarrow \operatorname{Im} d^{\,p-2} \longrightarrow \mathrm{E}^{p}(xs, y) \longrightarrow \mathrm{E}^{p}(x, y) \longrightarrow \operatorname{Ker} d^{\,p-1} \longrightarrow 0.$$

Then

$$\begin{aligned} e_p(x, y) - e_p(xs, y) &= \dim \operatorname{Ker} d^{p-1} - (e_{p-2}(xs, y/ys) - \dim \operatorname{Ker} d^{p-2}) \\ &\geq \dim \operatorname{Ker} \delta^{p-1} - (e_{p-2}(xs, y/ys) - \dim \operatorname{Ker} \delta^{p-2}) \\ &= e_{p-1}(xs, y) - e_{p-1}(xs, ys), \end{aligned}$$

where the last equality uses the exactness of the first row of (3.1.1). □

Corollary 3.3 [C, 3.9] *Suppose that $xs > x$, $ys < y$, and $xs \not< ys$. For all p,*

$$e_p(x, y) = e_p(xs, y) + e_{p-1}(xs, y).$$

Proof. Because $\mathrm{E}(xs, ys) = 0$, $\delta = 0$ and $d = 0$. The conditions for equality in 3.2 are satisfied. □

These results led naturally to the conjecture that $e_p = r_p$ for all p [C, 3.1]. It was soon discovered that there are examples where $r_p(x, y)$ is negative [Boe], so equality in 3.2 can not hold in general. One easy consequence of [GJ, 5.2.3] is that r_1 is, at least, a lower bound for e_1. (Later, it will be shown that $e_1 \neq r_1$.)

Proposition 3.4 $e_1 \geq r_1$

Proof. Assume there is a counterexample, $e_1(x, y) < r_1(x, y)$, with x maximal in the Bruhat ordering. If $x = w_0$, $e_1(w_0, y) = 0 = r_1(w_0, y)$ so $x < w_0$. Choose an $s \in S$ with $xs > x$. There are two cases to consider.

If $ys > y$, then $e_1(x, y) = e_1(xs, ys)$ by 3.1. Since x is maximal, $e_1(xs, ys) \geq r_1(xs, ys) = r_1(x, y)$ by equation (3.0.2).

If $ys < y$, then 3.2 implies that $e_1(x, y) \geq e_1(xs, y) + e_0(xs, y) - e_0(xs, ys)$. Since $e_0 = r_0$ and x is maximal, $e_1(x, y) \geq r_1(xs, y) + r_0(xs, y) - r_0(xs, ys) = r_1(x, y)$ by (3.0.1).

In either case, $e_1(x, y) \geq r_1(x, y)$, which contradicts the choice of x. □

The twisted sequence in diagram (3.1.1) has the same terms as the two-line spectral sequence of [C, 3.4]. It is an indirect resolution of the conjecture that the coboundary of the spectral sequence should factor as $d = \delta_1 \delta$ [C, p. 37]. It can also be substituted for the spectral sequence in many of the proofs. As an example, one result that is needed below will be re-proved here.

Proposition 3.5 [C, 3.8] *If* $x \le y$ *and* $n = \ell(x,y)$, *then* $e_n(x,y) = 1$.

Proof. Suppose that $x \le y$ and assume that there is a counterexample with x maximal. If $x = w_0$, then $y = w_0$, $n = 0$ and $e_0(w_0, w_0) = 1$ so $x < w_0$. Choose an $s \in S$ so that $xs > x$. There are two cases to consider.

If $ys > y$, and $e_n(x,y) = e_n(xs, ys)$ by 3.1. Because x is maximal and $xs \le ys$, $e_n(xs, ys) = 1$.

If $ys < y$, then consider diagram (3.1.1) with $\mathrm{E} = \mathrm{E}^{n-1}$ and apply the vanishing properties.

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}^{-}(xs, y/ys) & \xrightarrow{\delta} & 0 & \longrightarrow & \mathrm{E}(xs,y) & \longrightarrow & \mathrm{E}(xs, y/ys) & \longrightarrow\ 0 \\
 & \| & & \downarrow & & \delta_2 \downarrow & & \| & \\
\longrightarrow & \mathrm{E}^{-}(xs, y/ys) & \xrightarrow{d} & 0 & \longrightarrow & \mathrm{E}^{+}(x,y) & \longrightarrow & \mathrm{E}(xs, y/ys) & \longrightarrow\ 0.
\end{array}$$

Then δ_2 is an isomorphism, so $e_n(x,y) = e_{n-1}(xs,y)$. But $e_{n-1}(xs,y) = 1$ since $xs \le y$ and x is maximal.

In either case, $e_n(x,y) = 1$, which contradicts the choice of x. □

In the remainder of this section, the recursive calculation of $e_{n-1}(x,y)$ where $n = \ell(x,y)$ will be considered. Suppose that $x < xs < ys < y$ for some $s \in S$. Applying diagram (3.1.1) with $\mathrm{E} = \mathrm{E}^{n-2}$ yields

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}^{-}(xs, y/ys) & \xrightarrow{\delta} & \mathrm{E}(xs,ys) & \longrightarrow & \mathrm{E}(xs,y) & \longrightarrow & \mathrm{E}(xs, y/ys) & \longrightarrow\ 0 \\
 & \| & & \delta_1 \downarrow & & \downarrow & & \| & \\
\longrightarrow & \mathrm{E}^{-}(xs, y/ys) & \xrightarrow{d} & \mathrm{E}^{+}(xs,y) & \longrightarrow & \mathrm{E}^{+}(x,y) & \longrightarrow & \mathrm{E}(xs, y/ys) & \longrightarrow\ 0.
\end{array} \tag{3.5.1}$$

By 3.5, $e_{n-2}(xs, ys) = e_{n-1}(xs, y) = 1$ so that δ_1 is an isomorphism or zero. But δ_1 is part of the exact sequence

$$\mathrm{E}^{n-2}(xs, ys) \xrightarrow{\delta_1} \mathrm{E}^{n-1}(xs, y) \longrightarrow \mathrm{E}^{n-1}(M_{xs}, \theta M_y) \longrightarrow 0,$$

showing that δ_1 is an isomorphism, if and only if, $\mathrm{E}^{n-1}(M_{xs}, \theta M_y)$ is zero. By the adjoint pairing (T,R), $\mathrm{E}^{n-1}(M_{xs}, \theta M_y)$ is isomorphic to $\mathrm{E}^{n-1}(TM_{xs}, TM_y)$. The vanishing behavior of this singular extension group determines whether d is zero or surjective. This suggests a conjecture on singular vanishing.

Conjecture 3.6 *If* $x < xs < ys < y$, *then* $\mathrm{E}^{n-1}(TM_{xs}, TM_y) = 0$, *where* $n = \ell(x,y)$.

Proposition 3.7 *Suppose that* $x < y$ *and let* $n = \ell(x,y)$. *Conjecture 3.6 implies that*

$$e_{n-1}(x,y) = r_1(x,y).$$

Proof. Assume there is a counterexample with x maximal. Because $y \le w_0$, $x < w_0$ and there is an $s \in S$ with $xs > x$. There are three cases to consider.

If $ys > y$, 3.1 implies that $e_{n-1}(x,y) = e_{n-1}(xs, ys)$. Since x is maximal and $xs < ys$, $e_{n-1}(xs, ys) = r_1(xs, ys) = r_1(x,y)$ by equation (3.0.2).

If $ys < y$ and $xs \not< ys$, $e_{n-1}(x,y) = e_{n-1}(xs,y) + e_{n-2}(xs,y)$ by 3.3. Since $xs \le y$, $e_{n-1}(xs,y) = 1$ by 3.5. If $xs = y$, then $n = 1$ and $e_0(x,y) = r_0(x,y) = r_1(x,y)$ so $xs < y$ by the choice of x. Because x is maximal, $e_{n-2}(xs,y) = r_1(xs,y)$. Then $e_{n-1}(x,y) = 1 + r_1(xs,y) = r_1(x,y)$ by equation (3.0.2).

If $x < xs < ys < y$ and assuming that conjecture 3.6 is true, δ_1 in diagram (3.5.1) is an isomorphism. Then $e_{n-1}(x,y) = e_{n-2}(xs,y)$. Because x is maximal, $e_{n-2}(xs,y) = r_1(xs,y) = r_1(x,y)$ by equation (3.0.2).

In each case, $e_{n-1}(x,y) = r_1(x,y)$, which contradicts the choice of x. □

4. Applications in category $\mathcal{O}$: younger results

Most of the results of the last section have been known for a long time. The newer results involve r_1. The first new result in this direction was published by Mazorchuk in 2007.

Proposition 4.1 [Maz, Lemma 33] $e_1(1,w_0) = |S|$.

Corollary 4.2 *For all* $x, y \in W$,

(i) $e_1(x,w_0) = r_1(x,w_0)$ *and*
(ii) $e_1(1,y) = r_1(1,y)$.

The first item of 4.2 is equivalent to the original statement of [Maz, Theorem 32] (adjusting for anti-dominance and ignoring the grading). It is expressed here in terms of r_1. The proof of the corollary uses the following lemma.

Lemma 4.3 *Suppose that* $xs > x$ *and* $ys < y$ *for some* $s \in S$. *If* $e_1(x,y) = r_1(x,y)$, *then* $e_1(xs,y) = r_1(xs,y)$

Proof. Suppose that $e_1(xs,y) \ne r_1(xs,y)$. By 3.4, $e_1(xs,y) > r_1(xs,y)$. Using 3.2 and 3.0.1,

$$\begin{aligned} e_1(x,y) &\ge e_1(xs,y) + e_0(xs,y) - e_0(xs,ys) \\ &> r_1(xs,y) + r_0(xs,y) - r_0(xs,ys) = r_1(x,y), \end{aligned}$$

so $e_1(x,y) \ne r_1(x,y)$ □

Proof of the corollary. To show that $e_1(1,w_0) = r_1(1,w_0)$, apply [Hum1, 7.10(20)] with $x = 1$ and $w = w_0$ to get

$$\sum_{1 \le y \le w_0} R_{1,y} = q^n,$$

where $n = \ell(1,w_0)$. The coefficient of q^{n-1} on the left-hand side is

$$(-1)^1 r_{n-1}(1,w_0) + |S|,$$

so $r_1(1,w_0) = r_{n-1}(1,w_0) = |S|$.

To prove item (i), assume that there is a counterexample with x minimal. Then $x > 1$ and there is an $s \in S$ with $xs < x$. By minimality of x, $e_1(xs,w_0) = r_1(xs,w_0)$. The lemma implies that $e_1(x,w_0) = r_1(x,w_0)$, contradicting the choice of x.

The proof of item (ii) is similar. □

The next development was Noriyuki Abe's preprint that originally appeared on the ArXiv in 2010. Let $v(x,y)=e_1(x,y)-e_0(x,w_0/y)$ if $x\leq y$ and let $v(x,y)=0$ if $x\not\leq y$. If $x\leq y$, then $v(x,y)=\dim V(w_0x,w_0y)$ in Abe's notation. Then [Abe1, theorem 4.4] becomes $v=r_1$. As stated, the theorem is not true. There are 16 pairs (x,y) in type B_3 with $r_1(x,y)=4$ but, by definition, $v\leq 3$ [Abe1, Theorem 1.1(1)]. Abe's recursion for V [Abe1, Theorem 4.3] does imply that $v\leq r_1$ (by comparison with 3.0.2). Then, combined with 3.4, $v\leq r_1\leq e_1$ or

$$r_1(x,y)\leq e_1(x,y)\leq r_1(x,y)+e_0(x,w_0/y).$$

Note that $e_0(1,w_0/y)=0$ and $e_0(x,w_0/w_0)=0$, so Abe's inequality does generalize 4.2. Although $v\neq r_1$, Abe has communicated an example in type B_3 showing that $e_1\neq r_1$ [Abe2].

In the remainder, the twisted sequence approach will be used to prove properties of v that correspond with Abe's results from [Abe1].

PROPOSITION 4.4 *If $xs>x$ and $ys<y$, then $e_0(xs,w_0/y)=e_0(x,w_0/ys)$.*

Proof. Let $M=M_{xs}$ and $N=M_{w_0}/M_{ys}$. There is a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \theta M_{ys} & \longrightarrow & \theta M_{w_0} & \longrightarrow & \theta N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{ys} & \longrightarrow & M_{w_0} & \longrightarrow & N & \longrightarrow & 0. \end{array} \tag{4.4.1}$$

By the snake lemma, $KN=M_{w_0}/M_y$. By the adjoint pairing (C,K), $\mathrm{H}(M_{xs},KN)$ and $\mathrm{H}(M_x,N)$ are isomorphic. □

By 3.1, if $xs>x$ and $ys<y$, $e_1(xs,y)=e_1(x,ys)$ which proves the following property of v, which corresponds to [Abe1, 4.3(1)].

COROLLARY 4.5 *If $xs>x$ and $ys<y$, then $v(xs,y)=v(x,ys)$.*

Next, there is another ladder diagram that links extensions of fractional Verma modules to the twisted sequence.

PROPOSITION 4.6 *Suppose that $xs>x$ and $ys<y$. There is a commutative diagram with exact rows,*

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{E}^-(xs,y/ys) & \xrightarrow{\delta} & \mathrm{E}(x,w_0/ys) & \xrightarrow{\alpha} & \mathrm{E}(x,w_0/y) & \xrightarrow{\kappa} & \mathrm{E}(xs,y/ys) & \longrightarrow \\ & \| & & \downarrow{\scriptstyle\delta_1} & & \downarrow{\scriptstyle\delta_2} & & \| & \\ \longrightarrow & \mathrm{E}^-(xs,y/ys) & \xrightarrow{d} & \mathrm{E}^+(xs,y) & \xrightarrow{\gamma} & \mathrm{E}^+(x,y) & \xrightarrow{\chi} & \mathrm{E}(xs,y/ys) & \longrightarrow \end{array},$$

where the second row is the same as the second row of diagram (3.1.1).

Proof. The proof is similar in structure to the proof of 2.1. Fix a commuting triangle of Verma module injections,

$$\begin{array}{ccc} M_{ys} & \longrightarrow & M_y \\ \| & & \downarrow \\ M_{ys} & \longrightarrow & M_{w_0}. \end{array} \tag{4.6.1}$$

Diagram 1:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}^{-}(xs,y/ys) & \xrightarrow{\delta} & \mathrm{E}(x,w_0/ys) & \xrightarrow{\alpha} & \mathrm{E}(x,w_0/y) & \xrightarrow{\kappa} & \mathrm{E}(xs,y/ys) & \longrightarrow \\
& {\scriptstyle\delta_3}\downarrow & & \| & & \| & & {\scriptstyle\delta_3}\downarrow & \\
\longrightarrow & \mathrm{E}(x,y/ys) & \longrightarrow & \mathrm{E}(x,w_0/ys) & \longrightarrow & \mathrm{E}(x,w_0/y) & \longrightarrow & \mathrm{E}^{+}(x,y/ys) & \longrightarrow
\end{array}$$

The map δ_3 is the same as the isomorphism δ_3 from diagram (2.1.2) with $M = M_{xs}$ and $N = M_y$. The second row is the long exact sequence associated to the exact sequence,

$$M_y/M_{ys} \hookrightarrow M_{w_0}/M_{ys} \twoheadrightarrow M_{w_0}/M_y.$$

Define δ and κ so that the diagram commutes. This produces a commutative diagram with exact rows.

Diagram 2:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}(x,y/ys) & \longrightarrow & \mathrm{E}(x,w_0/ys) & \longrightarrow & \mathrm{E}(x,w_0/y) & \xrightarrow{\delta_7} & \mathrm{E}^{+}(x,y/ys) & \longrightarrow \\
& \| & & {\scriptstyle\delta_5}\downarrow & & {\scriptstyle\delta_6}\downarrow & & \| & \\
\longrightarrow & \mathrm{E}(x,y/ys) & \xrightarrow{\delta_4} & \mathrm{E}^{+}(x,ys) & \longrightarrow & \mathrm{E}^{+}(x,y) & \longrightarrow & \mathrm{E}^{+}(x,y/ys) & \longrightarrow
\end{array}$$

This is a commutative diagram with exact rows where δ_k, $4 \le k \le 7$ are natural connecting maps (all derived from rotations of diagram (4.6.1)). For example, the middle square commutes because of the short ladder,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{ys} & \longrightarrow & M_{w_0} & \longrightarrow & M_{w_0}/M_{ys} & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & M_y & \longrightarrow & M_{w_0} & \longrightarrow & M_{w_0}/M_y & \longrightarrow & 0.
\end{array}$$

Diagram 3:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}(x,y/ys) & \longrightarrow & \mathrm{E}^{+}(x,ys) & \longrightarrow & \mathrm{E}^{+}(x,y) & \longrightarrow & \mathrm{E}^{+}(x,y/ys) & \longrightarrow \\
& {\scriptstyle\delta_3}\uparrow & & {\scriptstyle\gamma'}\uparrow & & \| & & {\scriptstyle\delta_3}\uparrow & \\
\longrightarrow & \mathrm{E}^{-}(xs,y/ys) & \xrightarrow{d} & \mathrm{E}^{+}(xs,y) & \xrightarrow{\gamma} & \mathrm{E}^{+}(x,y) & \xrightarrow{\chi} & \mathrm{E}(xs,y/ys) & \longrightarrow
\end{array}$$

This is a commutative diagram with exact rows because it is diagram (2.1.3) with $M = M_{xs}$ and $N = M_y$. Since γ' is an isomorphism, assembling the diagrams completes the proof. □

Applying the same argument as in the proof of 3.2 yields the following inequality.

Proposition 4.7 *Suppose that $xs > x$ and $ys < y$. For all p,*

$$e_p(x,y) - e_p(xs,y) \ge e_{p-1}(x,w_0/y) - e_{p-1}(x,w_0/ys).$$

This is an equality if, and only if, $\operatorname{Ker} d^{p-1} = \operatorname{Ker} \delta^{p-1}$ *and* $\operatorname{Ker} d^{p-2} = \operatorname{Ker} \delta^{p-2}$.

Corollary 4.8 *If* $xs > x$ *and* $ys < y$*, then*

$$e_1(x,y) - e_1(xs,y) \geq e_0(x, w_0/y) - e_0(xs, w_0/y)$$

and this is an equality if, and only if, $\operatorname{Ker} d^0 = \operatorname{Ker} \delta^0$

Proof. Taking $p = 1$ in 4.7,

$$e_1(x,y) - e_1(xs,y) \geq e_0(x, w_0/y) - e_0(x, w_0/ys).$$

By 4.4, $e_0(x, w_0/ys) = e_0(xs, w_0/y)$. □

The conclusion is equivalent to $v(x,y) \geq v(xs,y)$. When $xs < ys$, Abe proves $v(x,y) = v(xs,y)$ by showing that the images of $\mathrm{E}^1(xs,y)$ and $\mathrm{E}^1(x,y)$ in $\mathrm{E}^1(x,w_0)$ are the same [Abe1, 4.3(2)].

The preceding proposition is sufficient, by itself, to explain Abe's counter-example for $e_1 = r_1$. In type B_3, let s_1, s_2, and s_3 be the simple root reflections, where s_1s_2 has order 3 and s_2s_3 has order 4. Take $x = s_1s_3$, $y = w_0s_3 = s_2s_3s_1s_2s_3s_2s_1s_2$, and $s = s_2$. Using the work of H. Matumoto [Mat] on scalar, generalized Verma module homomorphisms, Abe shows that there is a nonzero homomorphism between M_x and M_{w_0}/M_y so $e_0(x, w_0/y) \neq 0$ [Abe2]. Kazhdan-Lusztig multiplicities imply that $e_0(x, w_0/y) - e_0(xs, w_0/y) = 1$. By 4.8, $e_1(x,y) > e_1(xs,y)$, which means $e_1(x,y) \neq r_1(x,y)$.

Proposition 4.9 *Suppose that* $x < xs \leq y$ *and* $ys < y$*. If* $xs \not< ys$*, then* $v(x,y) \leq v(xs,y) + 1$ *and this is an equality if, and only if,* $\operatorname{Ker} \delta^0 = 0$.

Proof. In 4.6, $d = 0$ by 3.3. Also $e_0(xs, y/ys) = 1$ implies that $e_0(x, w_0/y) - e_0(x, w_0/ys) \leq 1$. □

The condition for equality in 4.9 must somehow be equivalent to the condition $v_s \notin sV(w_0xs, w_0y)$ from [Abe1, 4.3(2)]. Finally, another twisted sequence can be used to prove a result that is also consistent with [Abe1, 4.3(2)].

Suppose that $xs > x$ and $ys < y$. Let $M = M_{xs}$ and $N = M_{w_0}/M_{ys}$. There is a twisted sequence associated to N. From diagram (4.4.1), $IN = M_{w_0s}/M_{ys}$ so, by 2.1, there is a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{E}^{-}(xs, \frac{w_0}{w_0s}) & \xrightarrow{\delta} & \mathrm{E}(xs, \frac{w_0s}{ys}) & \xrightarrow{\alpha} & \mathrm{E}(xs, \frac{w_0}{ys}) & \xrightarrow{\kappa} & \mathrm{E}(xs, \frac{w_0}{w_0s}) & \longrightarrow \\
& \| & & \downarrow{\scriptstyle\delta_1} & & \downarrow{\scriptstyle\delta_2} & & \| & \\
\longrightarrow & \mathrm{E}^{-}(xs, \frac{w_0}{w_0s}) & \xrightarrow{d} & \mathrm{E}^{+}(xs, \frac{w_0}{y}) & \xrightarrow{\gamma} & \mathrm{E}^{+}(x, \frac{w_0}{ys}) & \xrightarrow{\chi} & \mathrm{E}(xs, \frac{w_0}{w_0s}) & \longrightarrow .
\end{array} \tag{4.9.1}$$

Proposition 4.10 *Suppose that* $x < xs \leq y$ *and* $ys < y$*. If* $xs \not< w_0s$*, then* $v(x,y) = v(xs,y) + 1$.

Proof. Because $ys < w_0s$, $xs \not< w_0s$ implies $xs \not< ys$ and hence $e_0(xs, w_0s/ys) = 0$. If E is identified with E^0 in diagram (4.9.1), κ is an injective map, which implies that δ_2 is injective. Working through the definitions, there is a commutative diagram,

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{H}(xs, y/ys) & \longrightarrow & \mathrm{H}(xs, w_0/ys) \\
& & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_2} \\
& & \mathrm{E}^1(x, w_0/ys) & = & \mathrm{E}^1(x, w_0/ys),
\end{array}$$

where δ is the homomorphism defined in the proof of 4.6. Since δ_2 is injective, $\operatorname{Ker}\delta = 0$ and $v(x, y) = v(xs, y) + 1$ by 4.9. $\square$

In a similar vein, one can prove that $v(x, y) = v(xs, y)$ if $x < xs < ys < y$ and $e_0(xs, w_0s/ys) = 0$. In that case, $e_1(x, y) = e_1(xs, y)$ as well.

If the goal is a general recursive formula for e_1, then the goal is well over the horizon. The classic conjecture, $e_1 = r_1$, is false. Abe's recursion for v is very effective (and v is bounded above by the rank of $\mathfrak{g}$), but the resulting determination of e_1 depends on the very difficult problem of generalized Verma module homomorphisms. If $x \le y$ and $e_0(x, w_0/y)$ is known, then $e_1(x, y) = v(x, y) + e_0(x, w_0/y)$.

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